Arch. Rational Mech. Anal. 199 (2011) 453–491
Digital Object Identifier (DOI) 10.1007/s00205-010-0392-9

On the Waterbag Continuum

Nicolas Besse

Communicated by C. Dafermos

Abstract

The aim of this paper is to study the existence of a classical solution for the waterbag model with a continuum of waterbags, which can been viewed as an infinite dimensional system of first-order conservation laws. The waterbag model, which constitutes a special class of exact weak solution of the Vlasov equation, is well known in plasma physics, and its applications in gyrokinetic theory and laser–plasma interaction are very promising. The proof of the existence of a continuum of regular waterbags relies on a generalized definition of hyperbolicity for an integrodifferential hyperbolic system of equations, some results in singular integral operators theory and harmonic analysis, Riemann–Hilbert boundary value problems and energy estimates.

1. The Vlasov equation and the waterbag model

Let x be the space variable with period $1, x \in \mathbb{R}/\mathbb{Z}$ and $v \in \mathbb{R}$ the velocity variable. Let $f(t, x, v)$ be the statistical distribution function of positive charged particles (ions) of a one-dimensional periodic quasi-neutral plasma. Then the distribution function f satisfies the Vlasov equation

$$\partial_t f + v\partial_x f + \frac{q_i}{m_i} E\partial_v f = 0, \tag{1}$$

with $q_i = Z_i e$, e the elementary charge, Z_i the number of charge, and where the electric field is given by $E = -\partial_x \phi$. The Vlasov equation is coupled to the quasi-neutral equation

$$\phi = \frac{k_B T_e}{n_{e0} e} \left(Z_i \int_{\mathbb{R}} f \mathrm{d}v - n_{e0} \right), \tag{2}$$

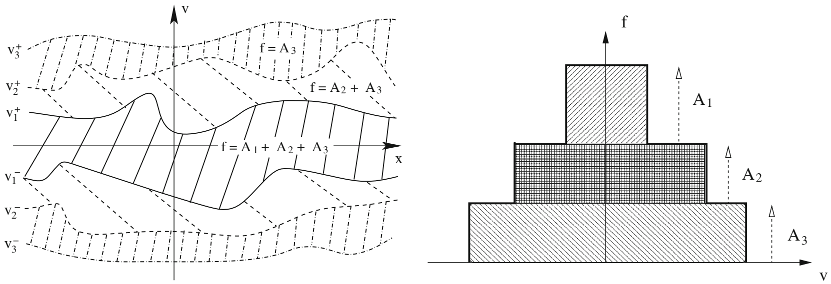

Fig. 1. Multiple waterbag: phase-space plot for a three-bag model (*left*) and corresponding MWB distribution function (*right*)

with k_B the Boltzmann constant, T_e the electrons' temperature, and where we have supposed that the electron density n_e follows the Maxwellian–Boltzmann distribution (adiabatic electrons) $n_{e0}\exp(e\phi/(k_B T_e))$ with $e\phi/(k_B T_e) \ll 1$. Let us note that the proof of classical or weak solutions of the system (1)–(2) seems difficult in its original form since the loss of spatial derivatives on the potential ϕ in equation (2) makes the classical Sobolev embedding theorems or averaging lemmas [10,28,36,37] useless for obtaining compactness.

Let us now consider $2\mathcal{N}$ non-closed contours in phase-space labelled v_j^+ and v_j^- (where $j = 1, \dots, \mathcal{N}$). Figure 1 shows the phase-space contours for a three-bag system ($\mathcal{N} = 3$) where the distribution function takes on three different constant values F_1, F_2 and F_3.

Introducing the bag heights $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ as also shown in Fig. 1, the distribution function reads

$$f(t,x,v) = \sum_{j=1}^{\mathcal{N}} \mathcal{A}_j \Big(\mathcal{H}(v_j^+(t,x) - v) - \mathcal{H}(v_j^-(t,x) - v)\Big), \tag{3}$$

where $\mathcal{H}$ is the Heaviside unit step function. Let us note that some of the parameters $\mathcal{A}_j$ can be negative. The function (3) is a solution of the Vlasov equation (1) in the sense of distribution theory, if and only if the set of following equations is satisfied

$$\partial_t v_j^\pm + v_j^\pm \partial_x v_j^\pm + \frac{q_i}{m_i}\partial_x \phi = 0, \quad j = 1 \dots \mathcal{N}. \tag{4}$$

Let us now introduce for each bag j the density n_j, average velocity u_j and pressure P_j such that $n_j = \mathcal{A}_j(v_j^+ - v_j^-)$, $u_j = (v_j^+ + v_j^-)/2$ and $p_j = mn_j^3/(12\mathcal{A}_j^2)$. For each bag j we recover the conservative form of the continuity and Euler equations (isentropic gas dynamics equations with $\gamma = 3$), namely

$$\partial_t n_j + \partial_x(n_j u_j) = 0, \tag{5}$$

$$\partial_t(n_j u_j) + \partial_x\left(n_j u_j^2 + \frac{p_j}{m_i}\right) + \frac{q_i}{m_i} n_j \partial_x \phi = 0. \tag{6}$$

The coupling between the bags is given by the total density $\sum_{j \leqq \mathcal{N}} n_j$ in the quasi-neutral equation

$$\phi = \frac{k_B T_e}{n_{e0} e} \left(Z_i \sum_{j=1}^{\mathcal{N}} n_j - n_{e0} \right). \tag{7}$$

The global entropic weak solution of the system (5)–(7) for one bag ($\mathcal{N} = 1$) has been proved in [18–20]. If all the parameters $\mathcal{A}_j$ are positive (single hump distribution function or unimodal function) then the existence of a local classical solution for the system constituted by equations (4) and (7) or the system (5)–(7) with a finite number of bags have been proved in [5]. Let us note that after a finite time, equations (4) and (7) or the system (5)–(7) will generate shocks, namely discontinuous gradients in x for $v_j^{\pm}$. Nevertheless the concept of entropic solution is not well suited here because the existence of an entropy inequality means that a scattering (or collision-like) process in velocity occurs on the right-hand side of the Vlasov equation.

This observation has been developed in the theory of kinetic formulation of scalar conservation laws. In fact, it was established in [12–14,35] that scalar conservation laws can be lifted as linear hyperbolic equations by introducing an extra variable $\xi \in \mathbb{R}$, which can be interpreted as a scalar momentum or velocity variable. The author of [14] proposed a numerical scheme, known as the transport-collapse method, to solve this linear kinetic equation. In fact, the solution of this numerical scheme can been seen as the solution of a variant version of the linear BGK kinetic model. The authors of [13,14,35] have proved, using BV estimates and Kruzhkov type analysis, that this numerical solution converges to the entropy solution of scalar conservation laws. This result was also shown in [59] using averaging lemmas [10,28,36,37] without bounded variation estimates. In [53] the authors also consider the BGK-like approximation, and again using BV estimates, they prove the convergence of the approximate solution to the right entropy solution when the relaxation time (the inverse of the collisional frequency) tends to zero. Right after, it was observed by the authors of [43,53] that, without any approximations, entropy solutions of scalar conservation laws can be directly formulated in kinetic style, known as kinetic formulation. Its generalization to systems of conservation laws seems impossible except for very peculiar systems [16,44,60]. On the right-hand side of these linear kinetic equations (the free streaming term) appear the velocity derivatives of nonnegative bounded measure, which is the signature of scattering (or diffusion) processes in velocity.

In order that the waterbag model should be equivalent to the Vlasov equation (without any diffusion-like term on the right-hand side of the Vlasov equation) we must consider multivalued solutions of the waterbag model beyond the first singularity. The appearance of a singularity (discontinuous gradients in x due to the Burgers term) is linked to the appearance of trapped particles which is characterized by the formation of vortexes and the development of the filamentation process in the phase-space. In special cases such as the study of nonlinear gyrokinetic turbulence in a cylinder [7,8], particles' dynamic properties [38] imply that the particles are not trapped but only passing through.

The waterbag representation of the distribution function, which is reminiscent of the Liouville geometric invariants, is not an approximation but rather a special class of initial conditions. Therefore, from the Liouville theorem (given the conservation of the measure and the connexity of an elementary volume of the phase-space), the waterbag solutions correspond to a special class of exact weak solution of the Vlasov equation. Introduced initially by DEPACKH [27], BERTRAND ET AL. [3,4,29] the waterbag model was shown to provide a bridge between fluid and kinetic description of a collisionless plasma, allowing us to keep the kinetic aspect of the problem with the same complexity as the multi-fluid model. In order to reduce the dimension of the phase-space (based on the existence of Liouville geometric invariants) this model has been recently reconsidered for gyrokinetic turbulence applications and laser–plasma interaction physics, with very fruitful results [5–9,48]. In fact, the system (1)–(2) (in the Vlasovian framework) or the system formed by the equations (4) and (7) (in the waterbag framework) are very crucial because they represent the parallel dynamic of particles subjected to a strong magnetic field as it occurs in magnetic controlled fusion devices (tokamak) where gyrokinetic turbulence governs the energy confinement time [7,8,48].

It is very interesting to notice the similarities between the waterbag model and the system of Benney equations which describes the dynamics of long shallow water waves and whose Hamiltonian structure has been intensively exploited [2,31,32,40–42,61,62]. Like the waterbag model, the Benney equations can be reduced to an infinite-dimensional system of first-order conservation laws, possessing an infinite number of integrals of motion. Each model can be derived from a special class of exact weak solutions of the Vlasov equation, through the Heaviside "closure" for the waterbag equations and the Dirac "closure" for the Benney equations.

2. The continuous waterbag model

Let x be the space variable with period 1, $x \in \mathbb{R}/\mathbb{Z}$. In order to consider an infinite number of bags we define two Lagrangian foliations to be the families of sheets $v^\pm = v^\pm(t,x,a)$ labelled by the Lagrangian label $a \in [0,1]$, where the waterbag continuum $v^\pm(t,x,a)$ comprises smooth functions such that $v^- \leqq v^+$, $\partial_a v^+ < 0$, $\partial_a v^- > 0$ and $\partial_x^\alpha v^+(a=1) = \partial_x^\alpha v^-(a=1)$, $\forall \alpha \leqq 1$. The equations (4) and (7) are still valid if we replace the counting measure by the Lebesgue measure da. In fact let us consider the distribution function

$$f(t,x,v) = \int_0^1 \big(\mathcal{H}(v^+(t,x,a)-v) - \mathcal{H}(v^-(t,x,a)-v)\big)\,\mathrm{d}\mu(a), \tag{8}$$

where

$$\mu(a) = \begin{cases} \mu^{\mathcal{N}}(a) = \sum_{j=1}^{\mathcal{N}} \mathcal{A}_j \delta(a-a_j), \\ \text{or} \\ \mu^\infty(a) = \mathbb{1}_{[0,1]}(a). \end{cases}$$

In addition we have $\mu^{\mathcal{N}} \rightharpoonup \mu^{\infty}$ for the weak-$*$ topology $\sigma(\mathcal{M}_b, \mathscr{C}_b)$ (topology of the narrow convergence) where $\mathcal{M}_b$ is the set of bounded Radon measures. Therefore it is easily verified by a direct check that f defined by equation (8) satisfies, in the distributional sense, the Vlasov equation

$$\partial_t f + v\partial_x f - \partial_x \phi \partial_v f = 0, \quad \phi = \int_{\mathbb{R}} f \mathrm{d}v, \tag{9}$$

if and only if the waterbag continuum $v^{\pm}$ satisfies the continuous waterbag model given by

$$\partial_t v^{\pm} + v^{\pm}\partial_x v^{\pm} + \partial_x \phi = 0, \tag{10}$$

$$\phi = \int_0^1 (v^+ - v^-)\mathrm{d}a. \tag{11}$$

If we set $c = v^+ - v- = \partial_a Z(t, x, a)$ $(\partial_a Z > 0)$ where

$$Z(t, x, a) = \int_0^a (v^+ - v^-)\mathrm{d}b, \tag{12}$$

and if we set $u = (v^+ + v^-)/2$, then (10)–(11) is equivalent to

$$\partial_t c + \partial_x (cu) = 0, \tag{13}$$

$$\partial_t u + u\partial_x u + \frac{c}{4}\partial_x c + \partial_x \left(\int_0^1 c\mathrm{d}a \right) = 0. \tag{14}$$

Let us note that in [15], the author proves the existence of a classical solution for the homogeneous hydrostatic equations rewritten as

$$\partial_t c + \partial_x (cu) = 0, \tag{15}$$

$$\partial_t u + u\partial_x u + \partial_x p = 0, \tag{16}$$

with p denoting the pressure, under the constraint

$$\int_0^1 c(t, x, a)\mathrm{d}a = 1. \tag{17}$$

The main difference between the system (15)–(17) and the system (13)–(14) is that, in the first one, the boundary is fixed by the constraint (17) whereas the second one is a free boundary problem.

3. Diagonalization of the continuous waterbag model

In this section we use the diagonalization and the hyperbolicity concepts developed in [56–58] for the integrodifferential system of the long-wave equation. In this framework, constraints which ensure the hyperbolicity of the waterbag continuum can be found while generalized eigenfunctions and eigenvalues can be computed explicitly.

3.1. Generalized eigenfunctions and eigenvalues

If we set $W = (u, c)^T$ and $\mathcal{A}$ the linear operator defined as, for $\varphi = (\varphi_1, \varphi_2)^T$,

$$\mathcal{A}\varphi = \begin{pmatrix} u\varphi_1 + \frac{c}{4}\varphi_2 + \int_0^1 \varphi_2 \mathrm{d}a \\ c\varphi_1 + u\varphi_2 \end{pmatrix},$$

then equations (13)–(14) can be recast in the following quasilinear system

$$\partial_t W + \mathcal{A}\partial_x W = 0. \tag{18}$$

The operator $\mathcal{A}$ is a bounded linear operator in $L^2([0, 1]) \times L^2([0, 1])$, for all fixed (t, x), which depends on $W(t, x, a)$. Let $\mathcal{A}^*$ be the adjoint operator relative to the inner product in $L^2([0, 1]) \times L^2([0, 1])$. A number $k^\alpha = k^\alpha(t, x)$ is called a characteristic number if we can associate to it a nontrivial generalized (in $\mathcal{D}'$, that is in the sense of distribution) vector-valued eigenfunction $\mathcal{F}^\alpha = (\mathcal{F}_1^\alpha, \mathcal{F}_2^\alpha)^T$ of the operator $\mathcal{A}^*$:

$$(\mathcal{A}^* - k^\alpha \mathcal{I})\mathcal{F}^\alpha = 0. \tag{19}$$

Generalized (in the distributional sense) eigenfunctions arise in restricting $\mathcal{A}$ to a Banach space $H \subset L^2([0, 1]) \times L^2([0, 1])$ and are elements of the space H^* dual to H relative to the bilinear form which is the inner product of $L^2([0, 1]) \times L^2([0, 1])$. A curve $x(t)$ is called a characteristic of the system (18) if

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k^\alpha(t, x), \tag{20}$$

where k^α is a real characteristic number or a real eigenvalue of $\mathcal{A}^*$. By applying $\mathcal{F}^\alpha$ to (18) we get

$$\begin{aligned} \langle \mathcal{F}^\alpha, W_t + \mathcal{A}W_x \rangle &= \langle \mathcal{F}^\alpha, W_t \rangle + \langle \mathcal{A}^* \mathcal{F}^\alpha, W_x \rangle \\ &= \langle \mathcal{F}^\alpha, W_t \rangle + \langle k^\alpha \mathcal{F}^\alpha, W_x \rangle \\ &= \langle \mathcal{F}^\alpha, W_t + k^\alpha W_x \rangle. \end{aligned} \tag{21}$$

The equality (21) is called a relation on the characteristics $x(t)$ defined by (20). In equation (21), the bilinear form $\langle \cdot, \cdot \rangle$ denotes the duality bracket between H and H^*. The system of equations (18) is called hyperbolic if all characteristic numbers determined by (19) are real and the system of relations on characteristics

$$\langle \mathcal{F}^\alpha, W_t + k^\alpha W_x \rangle = 0, \tag{22}$$

is equivalent to the system (18).

3.2. The generalized eigenvalue problem

Using (19) it follows, for all smooth vectorial function $\varphi = (\varphi_1, \varphi_2)^T$,

$$\langle \mathcal{F}^\alpha, (\mathcal{A} - k^\alpha \mathcal{I})\varphi \rangle = \left\langle \mathcal{F}_1^\alpha, u\varphi_1 - k^\alpha \varphi_1 + \frac{c}{4}\varphi_2 + \int_0^1 \varphi_2 \mathrm{d}a \right\rangle + \langle \mathcal{F}_2^\alpha, c\varphi_1 - k^\alpha \varphi_2 + u\varphi_2 \rangle.$$

As φ_1, and φ_2 are independent we get

$$\langle \mathcal{F}_1^\alpha, (u - k^\alpha)\varphi_1 \rangle + \langle \mathcal{F}_2^\alpha, c\varphi_1 \rangle = 0, \tag{23}$$

$$\left\langle \mathcal{F}_1^\alpha, \frac{c}{4}\varphi_2 + \int_0^1 \varphi_2 \mathrm{d}a \right\rangle + \langle \mathcal{F}_2^\alpha, -k^\alpha \varphi_2 + u\varphi_2 \rangle = 0. \tag{24}$$

Equation (23) becomes

$$\langle \mathcal{F}_2^\alpha, \varphi_1 \rangle = -\langle \mathcal{F}_1^\alpha, (u - k^\alpha) c^{-1} \varphi_1 \rangle, \tag{25}$$

while using (25), equation (24) becomes

$$\left\langle \mathcal{F}_1^\alpha, \left((u - k^\alpha)^2 - \frac{c^2}{4} \right) c^{-1} \varphi_2 \right\rangle - \int_0^1 \varphi_2 \mathrm{d}a \, \langle \mathcal{F}_1^\alpha, 1 \rangle = 0. \tag{26}$$

Therefore we have for all regular functions ϕ and ψ,

$$\langle \mathcal{F}_2^\alpha, \varphi \rangle = -\langle \mathcal{F}_1^\alpha, (u - k^\alpha) c^{-1} \varphi \rangle, \tag{27}$$

$$\langle \mathcal{F}_1^\alpha, (v^+ - k^\alpha)(v^- - k^\alpha) c^{-1} \psi \rangle - \int_0^1 \psi \mathrm{d}a \, \langle \mathcal{F}_1^\alpha, 1 \rangle = 0. \tag{28}$$

In order to build solutions for equation (28), we consider the set of numbers z in the complex plane outside the section of values of the waterbag continuum $v^\pm$, that is $z \neq v^\pm, \forall a \in [0, 1]$, where the variables (t, x) are considered as fixed. Therefore from (28) it follows

$$\langle \mathcal{F}_1^\alpha, \psi \rangle = \int_0^1 \frac{c\psi \mathrm{d}a}{(v^+ - z)(v^- - z)} \langle \mathcal{F}_1^\alpha, 1 \rangle. \tag{29}$$

Nontrivial solutions of (29) exist for $z = k^\alpha$ which satisfies the characteristic equation

$$\chi(z) = 1 - \int_0^1 \frac{c \mathrm{d}a}{(v^+ - z)(v^- - z)} = 0, \tag{30}$$

where we have imposed the normalization $\langle \mathcal{F}_1^\alpha, 1 \rangle = 1$. The characteristic equation (30) has two real roots $k^\pm$ on the real axis outside the domain of values of $v^\pm$, namely $[v_b^- = v^-(t, x, 0), v_b^+ = v^+(t, x, 0)]$. In fact, the derivative with respect to z of $\chi(z)$ is given by

$$\chi'(z) = 2 \int_0^1 \frac{(z - u) c da}{(v^+ - z)^2 (v^- - z)^2}. \tag{31}$$

We observe that χ' is strictly negative for $z \in]-\infty, v_b^-[$ and strictly positive for $z \in]v_b^+, +\infty[$. Moreover, the function $\chi(z) \to 1$ when $|z| \to \pm\infty$ and $\chi(z) \to -\infty$ when $z \to v_b^\pm$. Therefore there exists a root k^- (resp. k^+), belonging to the interval $]-\infty, v_b^-[$ (resp. $]v_b^+, +\infty[$) which is a first-order zero of $\chi(z)$. To summarize, we have for all x and t

$$-\infty < k^-(t,x) < \inf_a v^-(t,x,a), \tag{32}$$

$$\sup_a v^+(t,x,a) < k^+(t,x) < +\infty, \tag{33}$$

$$\chi(k^\pm(t,x)) = 0, \quad |\chi'(k^\pm(t,x))| > 0. \tag{34}$$

For the eigenvalues $z = k^\pm$ the action of the associated eigenfunctions $\mathcal{F}_i^\pm = \mathcal{F}_i^\pm(t,x,\nu)$, $i \in \{1,2\}$ on a regular test function ψ is given by

$$\langle \mathcal{F}_1^\pm, \psi \rangle = \int_0^1 \left(\frac{1}{v^-(\nu) - k^\pm} - \frac{1}{v^+(\nu) - k^\pm} \right) \psi(\nu) \mathrm{d}\nu, \tag{35}$$

$$\begin{aligned} \langle \mathcal{F}_2^\pm, \psi \rangle &= -\langle \mathcal{F}_1^\pm, (u - k^\pm) c^{-1} \psi \rangle \\ &= -\frac{1}{2} \int_0^1 \left(\frac{1}{v^-(\nu) - k^\pm} + \frac{1}{v^+(\nu) - k^\pm} \right) \psi(\nu) \mathrm{d}\nu. \end{aligned} \tag{36}$$

In the last equations (35) and (36) we have omitted the dependence of the function on the variables t and x to simplify the notation. We now define $k^{a,\pm}(t,x)$ such that it takes the value of the velocity $v^\pm(t,x,a)$ for a given a. For each value of $k^{a,\pm}(t,x)$ we define the distributions $\mathcal{F}_1^{a,\pm} = \mathcal{F}_1^{a,\pm}(t,x,\nu)$ such that

$$\mathcal{F}_1^{a,\pm} = \delta(\nu - a) + \mathcal{P}_{v^\pm(a)}(\nu), \tag{37}$$

where

$$\langle \mathcal{P}_{v^\pm(a)}, \psi \rangle = \mathrm{p.v.} \int_0^1 \left(\frac{1}{v^-(\nu) - v^\pm(a)} - \frac{1}{v^+(\nu) - v^\pm(a)} \right) (\psi(\nu) - \psi(a)) \mathrm{d}\nu. \tag{38}$$

A straightforward computation shows that if $k^\alpha = k^{a,\pm}(t,x)$ and $\mathcal{F}_1^\alpha = \mathcal{F}_1^{a,\pm}$ then equation (28) is satisfied and thus it means that $k^{a,\pm}$ are also eigenvalues of the linear operator $\mathcal{A}^*$, whose the associated eigenfunctions are $\mathcal{F}^{a,\pm} = (\mathcal{F}_1^{a,\pm}, \mathcal{F}_2^{a,\pm})^T$. While $\mathcal{F}_1^{a,\pm}$ is given by (37), the action of $\mathcal{F}_2^{a,\pm}$ on regular test function ψ is given by

$$\begin{aligned} \langle \mathcal{F}_2^{a,\pm}, \psi \rangle &= -\langle \mathcal{F}_1^{a,\pm}, (u - v^\pm(a)) c^{-1} \psi \rangle \\ &= \langle \mathcal{Q}_{v^\pm(a)}, \psi \rangle \pm \frac{1}{2} \langle \delta_a, \psi \rangle A(v^\pm(a)), \end{aligned}$$

where

$$\langle \mathcal{Q}_{v^\pm(a)}, \psi \rangle = -\frac{1}{2} \mathrm{p.v.} \int_0^1 \left(\frac{1}{v^-(\nu) - v^\pm(a)} + \frac{1}{v^+(\nu) - v^\pm(a)} \right) \psi(\nu) \mathrm{d}\nu, \tag{39}$$

$$A(v^\pm(a)) = 1 - \mathrm{p.v.} \int_0^1 \left(\frac{1}{v^-(\nu) - v^\pm(a)} - \frac{1}{v^+(\nu) - v^\pm(a)} \right) \mathrm{d}\nu. \tag{40}$$

3.3. Hyperbolicity of the waterbag continuum

3.3.1. Eigenvalues In this section we look for sufficient conditions which ensure that the continuous waterbag model has no more eigenvalues, and especially no imaginary eigenvalues, other than real eigenvalues $v^{\pm}$ and $k^{\pm}$. Let us define Γ^+ (resp. Γ^-) the curve lying in the interval $[v_b^-, v_c]$ (resp. $[v_c, v_b^+]$) oriented in the positive sense from v_b^- to v_c (resp. from v_c to v_b^+) where $v_c = v^-(a = 1) = v^+(a = 1)$. We set $\Gamma = \Gamma^- \cup \Gamma^+$.

If we set v such that $v_{|\Gamma^\pm} = v^{\pm}$, ω such that $\omega_{|\Gamma^\pm} = \omega^{\pm} = \partial_a v^{\pm}$, and γ such that $\gamma_{|\Gamma^\pm} = \gamma^{\pm} = 1/\omega^{\pm}$, then the function $\chi(z)$, can be recast as

$$\chi(z) = 1 - \int_\Gamma \frac{\gamma \, \mathrm{d}v}{v - z}, \quad \forall z \in \mathbb{C} \setminus \Gamma, \tag{41}$$

for v smooth enough (see Section 4.2). We denote by $\mathcal{C}(z, r)$ the circle of center z and radius r. Let us now consider the circle $\mathcal{C}(k^-, r_\delta)$ (resp. $\mathcal{C}(k^+, r_\delta)$) which bounds the region $S^-_{1,\delta}$ (resp. $S^-_{7,\delta}$). We also consider the circle $\mathcal{C}(v_b^-, r_\delta)$ (resp. $\mathcal{C}(v_b^+, r_\delta)$) which bounds the region $S^-_{2,\delta}$ (resp. $S^-_{6,\delta}$). Let $\mathcal{C}(v_c, r_\delta)$ be the circle which bounds the region $S^-_{4,\delta}$. Let us now consider a point $v_*^- \in]v_b^-, v_c[$ (resp. $v_*^+ \in]v_c, v_b^+[$) and the circle $\mathcal{C}(v_*^-, r_\delta)$ (resp. $\mathcal{C}(v_*^+, r_\delta)$) which bounds the region $S^-_{3,\delta}$ (resp. $S^-_{5,\delta}$). If we now suppose that the zero of the complex plane belongs to Γ, we denote by $\mathcal{E}_\delta$ the ellipse of zero-center and variable radius $R_\delta(z)$ for $z \in \mathcal{E}_\delta$ which contains the regions $\cup_{i=1}^7 S^-_{i,\delta}$. We define the open contours $\Gamma^-_{1,\delta}$, $\Gamma^-_{2,\delta}$, $\Gamma^+_{1,\delta}$ and $\Gamma^+_{2,\delta}$, lying in Γ, and connecting respectively circles $\mathcal{C}(v_b^-, r_\delta)$ to $\mathcal{C}(v_*^-, r_\delta)$, $\mathcal{C}(v_*^-, r_\delta)$ to $\mathcal{C}(v_c, r_\delta)$, $\mathcal{C}(v_c, r_\delta)$ to $\mathcal{C}(v_*^+, r_\delta)$ and $\mathcal{C}(v_*^+, r_\delta)$ to $\mathcal{C}(v_b^+, r_\delta)$. We then set $\mathscr{C}_\delta = \mathcal{C}_\delta \cup \mathcal{E}_\delta \cup \Gamma_\delta$ where $\Gamma_\delta = \cup_{i \in \{1,2\},\, j \in \{-,+\}} \Gamma^j_{i,\delta}$, and $\mathcal{C}_\delta = \cup_{i \in \{-,+\}} \mathcal{C}(k^i, r_\delta) \cup \mathcal{C}(v_b^i, r_\delta) \cup \mathcal{C}(v_*^i, r_\delta) \cup \mathcal{C}(v_c, r_\delta)$. Finally, we define $S^-_\delta = \cup_{i=0}^7 S^-_{i,\delta}$ and consider the region S^+_δ of the complex plane, bounded by the contour $\mathscr{C}_\delta$ (see Fig. 2). When $\delta \to 0$ we get $r_\delta \to 0$, $R_\delta \to \infty$, $S^+_\delta \to \mathbb{C} \setminus \Gamma \cup \{k^-, k^+\}$, $\Gamma_\delta \to \Gamma$ and $\mathscr{C}_\delta \to \Gamma$. Using the Sokhotskii–Plemelj formula [30,52,54], the limiting values of the analytic function $\chi(z)$ from the upper and lower half-planes in the section $\Gamma = [v_b^-, v_b^+]$ are given by

$$\chi^{\pm}(z) = 1 - \text{p.v.} \int_\Gamma \frac{\gamma \, \mathrm{d}v}{v - z} \mp i\pi\gamma, \quad \forall z \in \Gamma. \tag{42}$$

We will see further that it is necessary to suppose that the condition

$$\operatorname{ess} \inf_{z \in \Gamma} |\chi^{\pm}(z)| > 0 \tag{43}$$

holds. Let us note that the previous condition (43) is guaranteed as long as $\operatorname{ess} \inf_{z \in \Gamma} |\gamma| > 0$. According to the argument principle [1,30], we have

$$\Delta_{\mathscr{C}_\delta} \arg \chi(z) = \mathcal{Z} - \mathcal{P}, \tag{44}$$

where $\Delta_{\mathscr{C}_\delta} \arg \chi$ denotes the variation of the argument of the function χ along the curve $\mathscr{C}_\delta$, $\mathcal{Z}$ is the number of zeroes and $\mathcal{P}$ is the number of poles of the function χ in the region S^+_δ, bounded by the circuit $\mathscr{C}_\delta$, their multiplicities being taken into

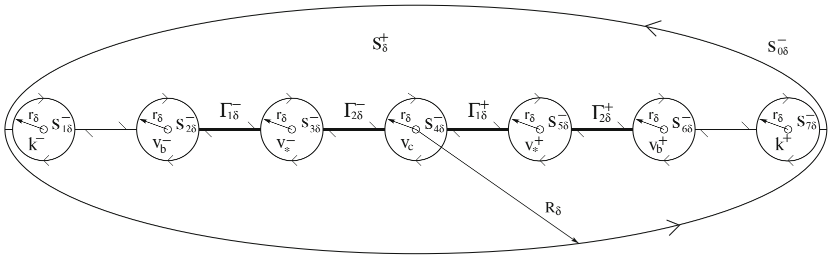

Fig. 2. The complex plane

account. Let us note that when we compute the quantity $\Delta_{\mathscr{C}_\delta} \arg \chi$, the open contour Γ_δ is traversed in one sense and its opposite (see Fig. 2). Since the function χ is analytic and does not have poles outside Γ, the condition $\Delta_{\mathscr{C}_\delta} \arg \chi = 0$ guarantees the absence of complex zeroes in the region S_δ^+.

Let us now take a look at the condition $\Delta_{\mathscr{C}_\delta} \arg \chi = 0$. First we observe that in the neighbourhood of the point $v_b^\pm$ and v_c, the expression (40) $(A(z),\ z \in \Gamma)$ can be expressed respectively in the form

$$\begin{aligned} A(z) &= \Omega_b^\pm(z) \mp \gamma^\pm(v_b^\pm \mp 0) \log(z - v_b^\pm), \\ A(z) &= \Omega_c(z) - \left\{\gamma^-(v_c - 0) - \gamma^+(v_c + 0)\right\} \log(z - v_c), \end{aligned} \tag{45}$$

where the functions $\Omega_b^\pm$ and Ω_c are bounded on $\mathbb{C}$ as long as for all t and x, $\omega(t, x, \cdot)$ and $\gamma(t, x, \cdot)$ belong to $\mathbb{P}\mathscr{C}^{0,\mu}(\Gamma)$ (the set of piecewise μ-Hölder continuous functions on Γ, with Hölder exponent μ) with $0 < \mu < 1$. From estimates (45) we deduce that there exists at least a point $v_*^- \in \Gamma^-$ (resp. $v_*^+ \in \Gamma^+$) such that $A(v_*^-) = 0$ (resp. $A(v_*^+) = 0$). Let us note that, in fact, the number of zeroes of $A(z)$ on Γ^- (resp. Γ^+) is odd. Nevertheless, without loss of generality, as we will see further, we can consider only one zero since the effects of an even number of zeroes on the argument variation of the function χ cancel by considering pairs of zeroes. Let us consider the variation of the argument of χ along the circle $\mathcal{C}(k^\pm, r_\delta)$. Since $\operatorname{sign}(\chi(k^\alpha + \sigma\varepsilon)) = \operatorname{sign}(\alpha\sigma)$, with $0 < \varepsilon \ll 1, \alpha = \pm 1$ and $\sigma = \pm 1$, $|\operatorname{Re}\chi(z)| \ll 1$ for z in the neighbourhood of $k^\pm$, and $\chi(k^\alpha + i\sigma\varepsilon) = A(k^\alpha) - i\sigma\varepsilon \int_\Gamma (\tau - k^\alpha)^{-2} \gamma \, d\tau + \mathcal{O}(\varepsilon^2)$, the variation of the argument of the function χ along the circle $\mathcal{C}(k^-, r_\delta)$ (resp. $\mathcal{C}(k^+, r_\delta)$) is equal to -2π (resp. -2π). Let us now consider the variation of the argument of χ along the circle $\mathcal{C}(v_*^\pm, r_\delta)$. Since $\operatorname{sign}\left(A(v_*^\alpha + \sigma\varepsilon)\right) = -\operatorname{sign}(\alpha\sigma)$, $A(v_*^\alpha + \sigma 0) = 0^{-\operatorname{sign}(\alpha\sigma)}$, $\gamma^\pm \neq 0$, and $\operatorname{sign}(\gamma^\pm) = \mp$, then using (42), the variation of the argument of the function χ along the circle $\mathcal{C}(v_*^-, r_\delta)$ (resp. $\mathcal{C}(v_*^+, r_\delta)$) is equal to 2π (resp. 2π). If we had considered an even number of roots, using the properties of the functions A, γ and (42), the variation of the argument of the function χ along small circles centered on these roots would be, in sum, equal to zero. Let us now consider the variation of the argument of χ along the circle $\mathcal{C}(v_b^\pm, r_\delta)$. Since in the neighbourhood of the points $v_b^\pm$, $\operatorname{Re}\chi \ll 0$, $\gamma^\pm \neq 0$ and $\operatorname{sign}(\gamma^\pm) = \mp$, then using (42), the variation of the argument of the function χ along the circle $\mathcal{C}(v_b^-, r_\delta)$ (resp. $\mathcal{C}(v_b^+, r_\delta)$) is equal to zero (resp. zero). Finally let us consider the variation of the argument of χ along the circle $\mathcal{C}(v_c, r_\delta)$. Since in the neighbourhood of the point v_c, $\mathrm{Re}\chi \gg 0$, $\gamma^\pm \neq 0$ and $\mathrm{sign}(\gamma^\pm) = \mp$, then using (42), the variation of the argument of the function χ along the circle $\mathcal{C}(v_c, r_\delta)$ is also equal to zero. To sum up, the variation of the argument of the function χ along the circles $\mathcal{C}_\delta$ is, in sum, equal to zero. Now, since $\chi(\infty) = 1$, we get $\Delta_{\mathcal{E}_\delta} \arg \chi \longrightarrow 0$ by passing to the limit as $\delta \longrightarrow 0$. Therefore, since the condition (43) holds, by passing to the limit as $\delta \longrightarrow 0$ the condition $\Delta_{\mathscr{C}_\delta} \arg \chi = 0$ finally becomes

$$\Delta_\Gamma \arg \frac{\chi^+}{\chi^-} = 4\pi. \tag{46}$$

We will see further that the conditions (43) and (46) are, in fact, solvability conditions for the Riemann–Hilbert boundary value problem or, equivalently, conditions for the Fredholmness and right-invertibility of a singular integral operator. Since the condition (43) holds, we can define $\mathcal{G}(z) = \chi^+(z)/\chi^-(z)$ and $\mathcal{G}^{-1}(z) = \chi^-(z)/\chi^+(z)$. Let us note that $\mathcal{G} = \exp(i\Theta(z))$ since $|\mathcal{G}(z)| = 1$ and $\arg \mathcal{G}(z) = -i \log \mathcal{G}(z) = \Theta(z)$. Moreover we have $\mathcal{G}(v_b^\pm \mp 0) = \mathcal{G}(v_c \pm 0) = 1$ and $\mathcal{G}(v_*^\pm - 0) = \mathcal{G}(v_*^\pm + 0) = -1$. The functions $\arg \mathcal{G}$ and $\mathcal{G}$ belong to the space $\mathscr{C}(\Gamma)$, the set of continuous functions on Γ. Let us set Γ' an open Liapunov curve (a Lipschitz Jordan curves) which does not cross the open Liapunov curve Γ such that that $\Gamma \cup \Gamma'$ constitutes a closed Liapunov curve and where the domain enclosed by the contour $\Gamma \cup \Gamma'$ does not contain the points $k^\pm$. Since $\mathcal{G}(v_b^\pm \mp 0) = 1$, we can extend continuously $\mathcal{G}$ on $\Gamma \cup \Gamma'$ by setting $\mathcal{G} = 1$ on Γ'. Therefore we can define the index of the function $\mathcal{G} \in \mathscr{C}(\Gamma)$, noted $\mathrm{Ind}(\mathcal{G})$ by $\mathrm{Ind}(\mathcal{G}) := \frac{1}{2\pi}\Delta_{\Gamma\cup\Gamma'} \arg \mathcal{G} = \frac{1}{2\pi}\Delta_\Gamma \arg \mathcal{G}$, which is equal to the winding number of the plane curve $\mathscr{R}(\mathcal{G})$ resulting from the range of the function $\mathcal{G}$ in the complex-plane. Let us note that $\mathrm{Ind}(\mathcal{G}) = -\mathrm{Ind}(\mathcal{G}^{-1})$. For more information about the definition and the properties of the index of a complex-variable function, we refer the reader to references [30,33,34,47,52,54].

To summarize, we have proven the following result

Lemma 1. *Let us assume that for all t and x, $\omega(t, x, .)$ and $\gamma(t, x, .)$ belong to $\mathbb{P}\mathscr{C}^{0,\mu}(\Gamma)$ with $0 < \mu < 1$. If v satisfy the conditions*

$$\mathrm{ess} \inf_{z\in\Gamma} |\chi^\pm(z)| > 0, \quad \textit{and} \quad \mathrm{Ind}(\mathcal{G}) = 2, \tag{47}$$

with $\mathcal{G}(z) = \chi^+(z)/\chi^-(z)$ and where $\chi^\pm$ are defined by (42), then the operator $\mathcal{A}$ has only the real eigenvalues $\{k^\pm, v^\pm\}$.

Remark 1. If we suppose that the smooth enough (typically Hölder-regularity) initial waterbag continuum satisfies for all x, the conditions $v^- \leqq v^+$, $\partial_a v^+ < 0$, $\partial_a v^- > 0$, $\partial_x^\alpha v^+(a = 1) = \partial_x^\alpha v^-(a = 1)$, $\forall \alpha \leqq 1$, and in addition fulfill strict convexity ($\partial_a^2 v^\pm > 0$) or concavity ($\partial_a^2 v^\pm < 0$) properties, then it can be checked that the conditions (47) hold.

3.3.2. The diagonal system We have shown that the eigenvalues of the continuous waterbag are reals under some conditions summarized in the Lemma 1. In order to show that the system is hyperbolic we must prove that the equations (22) and (18) are equivalent. Let us define the vector $\psi = (\psi_1, \psi_2)^T$ as

$$\begin{aligned}\psi_1 &= \partial_t u + u\partial_x u + \frac{c}{4}\partial_x c + \partial_x\left(\int_0^1 c\mathrm{d}a\right),\\ \psi_2 &= \partial_t c + \partial_x(cu).\end{aligned}$$

After some algebra we get

$$\begin{aligned}\langle \mathcal{F}^{a,\pm}, \psi\rangle &= \partial_t \mathcal{R}^\pm + v^\pm \partial_x \mathcal{R}^\pm,\\ \langle \mathcal{F}^{\pm}, \psi\rangle &= \partial_t r^\pm + k^\pm \partial_x r^\pm,\end{aligned}$$

where

$$\mathcal{R}^\pm(t,x,a) = v^\pm(t,x,a) + \int_0^1 \ln\left|\frac{v^\pm(a) - v^-(\nu)}{v^\pm(a) - v^+(\nu)}\right| \mathrm{d}\nu, \tag{48}$$

$$r^\pm(t,x) = k^\pm(t,x) + \int_0^1 \ln\left|\frac{k^\pm - v^-(\nu)}{k^\pm - v^+(\nu)}\right| \mathrm{d}\nu. \tag{49}$$

Therefore $\psi = 0$ implies the set of equations

$$\partial_t \mathcal{R}^\pm + v^\pm \partial_x \mathcal{R}^\pm = 0, \tag{50}$$

$$\partial_t r^\pm + k^\pm \partial_x r^\pm = 0. \tag{51}$$

As we have shown that (18) implies (22), let us now show that (22) implies (18). Let us suppose that we have $\langle \mathcal{F}^{a,\pm}, \psi\rangle = 0$ and $\langle \mathcal{F}^{\pm}, \psi\rangle = 0$, then we first get

$$\begin{aligned}\langle \mathcal{F}^{a,\pm}, \psi\rangle &= \langle \mathcal{F}_1^{a,\pm}, \psi_1\rangle + \langle \mathcal{F}_2^{a,\pm}, \psi_2\rangle\\ &= \left(\psi_1 \pm \frac{\psi_2}{2}\right) A(v^\pm(a))\\ &\quad + \text{p.v.} \int_0^1 \left(\psi_1 - \frac{\psi_2}{2}\right) \frac{\mathrm{d}\nu}{v^-(\nu) - v^\pm(a)}\\ &\quad - \text{p.v.} \int_0^1 \left(\psi_1 + \frac{\psi_2}{2}\right) \frac{\mathrm{d}\nu}{v^+(\nu) - v^\pm(a)}.\end{aligned} \tag{52}$$

If we set $\psi^\pm = \psi_1 \pm \frac{\psi_2}{2}$, we obtain

$$\langle \mathcal{F}^{a,\pm}, \psi\rangle = A(v^\pm(a))\psi^\pm + \text{p.v.} \int_{\Gamma^-} \frac{\gamma^-\psi^-\mathrm{d}v^-}{(v^- - v^\pm(a))} + \text{p.v.} \int_{\Gamma^+} \frac{\gamma^+\psi^+\mathrm{d}v^+}{(v^+ - v^\pm(a))}.$$

If we define Ψ such that $\Psi_{|_{\Gamma^\pm}} = \psi^\pm$ therefore the condition $\langle \mathcal{F}^{a,\pm}, \psi\rangle = 0$ is equivalent to

$$A(v(a))\Psi + \text{p.v.} \int_\Gamma \frac{\gamma\, \Psi\, \mathrm{d}\tau}{\tau - v(a)} = 0. \tag{53}$$

In the same way, the condition $\langle \mathcal{F}^{\pm}, \psi \rangle = 0$ is equivalent to

$$A(k)\Psi + \int_{\Gamma} \frac{\gamma\, \Psi\, \mathrm{d}\tau}{\tau - k} = 0, \tag{54}$$

where k denotes k^- or k^+. Therefore, singular integral equations (53) and (54) can be summarized as

$$A(z)\Psi + \text{p.v.} \int_{\Gamma} \frac{\gamma\, \Psi\, \mathrm{d}\tau}{\tau - z} = 0, \quad z \in \Gamma \cup \{k^{\pm}\}. \tag{55}$$

The singular integral equation (55) can be recast as

$$(\mathscr{P}_+\chi^-\mathcal{I} + \mathscr{P}_-\chi^+\mathcal{I})\Psi = 0, \tag{56}$$

with $\mathscr{P}_{\pm} = (\mathcal{I} \pm \mathcal{C}_{\Gamma})/2$, where the operator $\mathcal{C}_{\Gamma}$ is the singular Cauchy integral

$$(\mathcal{C}_{\Gamma}\varphi)(z) = \text{p.v.} \frac{1}{i\pi} \int_{\Gamma} \frac{\varphi(\tau)\mathrm{d}\tau}{\tau - z}.$$

Using the notations of the Theorem 3.1 of chapter 10 in [34] (with $p = 2$), and since $\mathcal{G} \in \mathscr{C}(\Gamma)$, we get $\delta(v_b^{\pm}) = \delta(v_c) = \delta(v_*^{\pm}) = \pi$ which leads to $\beta(v_b^{\pm}) = \beta(v_c) = \beta(v_*^{\pm}) = 0$ and thus $\rho = 1$. Moreover since $\text{Ind}(\mathcal{G}^{-1}) = -\text{Ind}(\mathcal{G})$, under the condition of Lemma 1, the Theorem 3.1 of chapter 10 (or Theorem 4.1 of chapter 9) in [34] holds and the operator $\mathscr{P}_+\chi^-\mathcal{I} + \mathscr{P}_-\chi^+\mathcal{I}$ is a Fredholm right-invertible operator on the weighted (with weight ρ) L^2-space $L^2_{\rho}(\Gamma)$ which is a Banach space equipped with the norm $\|\varphi\|_{L^2_{\rho}(\Gamma)} = \|\rho^{1/2}\varphi\|_{L^2(\Gamma)}$. The condition (43) ensures the Fredholmness of the singular integral operator $\mathscr{P}_+\chi^-\mathcal{I} + \mathscr{P}_-\chi^+\mathcal{I}$, while the condition $\text{Ind}(\mathcal{G}) = 2$ guarantees its right-invertibility. Moreover we get $\dim\ker(\mathscr{P}_+\chi^-\mathcal{I} + \mathscr{P}_-\chi^+\mathcal{I}) = -\text{Ind}(\mathcal{G}^{-1}) = 2$, while we have $\dim \text{coker}(\mathscr{P}_+\chi^-\mathcal{I} + \mathscr{P}_-\chi^+\mathcal{I}) = 0$. In order to show that the system (10)–(11) or (13)–(14) is equivalent to (50)–(51), we need to show that the dimension of the kernel of the singular integral operator $\mathscr{P}_+\chi^-\mathcal{I} + \mathscr{P}_-\chi^+\mathcal{I}$, can be reduced to zero. To this aim we first explicitly compute a base for the kernel of the singular integral operator (53), by solving directly the Riemann–Hilbert boundary value problem [30,33,34,52,54] associated with (53), and afterwards we show that $\ker(\mathscr{P}_+\chi^-\mathcal{I} + \mathscr{P}_-\chi^+\mathcal{I}) = \emptyset$, by using (54) and properties (34) of the eigenvalue problem. Let us define the sectionally analytic function

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{B\Psi\, \mathrm{d}\tau}{\tau - z}, \quad z \in \mathbb{C} \setminus \Gamma, \tag{57}$$

with $\Phi(\infty) = 0$, Ψ being the function appearing in the equation (56) and $B = i\pi\gamma \neq 0$. Using the Sokhotskii–Plemelj formula [30,52,54], the limiting values of the analytic function $\Phi(z)$ from the upper and lower half-planes in the section Γ are given by

$$\Phi^{\pm}(z) = \pm\frac{1}{2}B(z)\Psi(z) + \text{p.v.} \frac{1}{2\pi i} \int_{\Gamma} \frac{B(\tau)\Psi(\tau)\, \mathrm{d}\tau}{\tau - z}, \quad z \in \Gamma. \tag{58}$$

Using (58), equation (56) is equivalent to equation $\Phi^+(z) = \mathcal{G}(z)\Phi^-(z)$, $z \in \Gamma$. Therefore we consider the following homogeneous Riemann–Hilbert boundary value problem which consists in finding the function $\Phi(z)$, sectionally holomorphic in $\mathbb{C} \setminus \Gamma$, whose boundary values Φ^+ and Φ^- satisfy the boundary condition

$$\Phi^+(z) = \mathcal{G}(z)\Phi^-(z), \quad z \in \Gamma. \tag{59}$$

Since $\mathrm{Ind}(\mathcal{G}) = 2$ and $k^\pm$ are first-order zeroes of χ, then the function $\mathcal{X} = \chi/\varrho$, where $\varrho = (z - k^-)(z - k^+)$, is a canonical solution [30,52,54] of the Riemann–Hilbert boundary value problem (59), that is, a sectionally analytic function satisfying the boundary condition (59), having zero order everywhere in the finite part of the complex-plane (except possibly at some finite numbers of points of the contour Γ, called ends, where the function is bounded or has an infinity of integrable order) and such that the point at infinity has an order equal to $-\mathrm{Ind}(\mathcal{G})$, that is, minus the index of the Riemann–Hilbert boundary value problem (59). Therefore the problem (59) can be reinterpreted as finding the function $\Xi(z)$, sectionally holomorphic in $\mathbb{C} \setminus \Gamma$, whose boundary values $\Xi^\pm$ satisfy the boundary condition $\Xi^+(z) = \Xi^-(z)$, $z \in \Gamma$, where $\Xi = \Phi/\mathcal{X}$. The last boundary condition indicates that the function Ξ^+ analytic in the upper plane and the function Ξ^- analytic in the lower plane constitute the analytic continuation of each other through the contour Γ. According to the generalized Liouville Theorem [1,30], the function Ξ must reduce to a second-degree polynomial $P_2(z)$ with arbitrary coefficients, that is, $\Phi = \mathcal{X} P_2$. But, since $\Phi(\infty) = 0$ and $\chi(\infty) = 1$, the coefficient of the monomial of highest degree is zero, hence $P_2(z)$ reduces to $P_1(z)$, a polynomial of degree one. Therefore there exist two constants α_0 and α_1 such that $\Phi = (\alpha_0 + \alpha_1 z)\chi(z)/\varrho(z)$. Since $B\Psi = \Phi^+ - \Phi^-$ and $B \neq 0$, using the Sokhotskii–Plemelj formula [30,52,54], we obtain

$$\Psi = -2\left(\frac{C_0}{z - k^-} + \frac{C_1}{z - k^+}\right), \tag{60}$$

where $C_0 = -(\alpha_1 k^- + \alpha_0)/(k^+ - k^-)$ and $C_1 = (\alpha_1 k^+ + \alpha_0)/(k^+ - k^-)$. If we now plug (60) into equation (54) we obtain the linear system of equations $\mathcal{M}\mathcal{C} = 0$, where $\mathcal{C} = (C_0, C_1)^T$, and $\mathcal{M}$ is a symmetric matrix with the entries $\mathcal{M}_{11} = \mathcal{M}_{--}$, $\mathcal{M}_{22} = \mathcal{M}_{++}$, $\mathcal{M}_{12} = \mathcal{M}_{-+}$, $\mathcal{M}_{21} = \mathcal{M}_{+-}$, such that

$$\mathcal{M}_{ij} = -2\int_\Gamma \frac{\gamma\, \mathrm{d}\tau}{(\tau - k^i)(\tau - k^j)}, \quad i, j \in \{-, +\}.$$

From properties (34), we get $\mathcal{M}_{11} = 2\chi'(k^-) < 0$ and $\mathcal{M}_{22} = 2\chi'(k^+) > 0$ so that $\det \mathcal{M} < 0$. Therefore $\mathcal{M}$ is invertible, which implies that $\Psi = 0$ in $L^2(\Gamma)$, and thus $\psi = 0$ almost everywhere.

Finally, we have proven that the systems (10)–(11), (13)–(14) and (50)–(51) are hyperbolic with real eigenvalues $\{v^\pm, k^\pm\}$ and equivalent to each other. The new variables $\mathcal{R}^\pm$ and $r^\pm$ are call Riemann invariants and are constant along the characteristics $\mathrm{d}x^{a,\pm}(t)/\mathrm{d}t = v^\pm(t, x^{a,\pm}(t), a)$ and $\mathrm{d}x^\pm(t)/\mathrm{d}t = k^\pm(t, x^\pm(t))$ respectively.

4. Existence and uniqueness of a classical solution

4.1. Notations

We first note $\|\cdot\|_{L^p_{xa}}$ (resp. $\|\cdot\|_{L^p_x}$), with $1 \leqq p \leqq \infty$ the Lebesgue L^p-norm with respect to $x \in \mathbb{R}/\mathbb{Z}$, $a \in [0, 1]$ (resp. $x \in \mathbb{R}/\mathbb{Z}$). We also note $\|\cdot\|_{W^{m,p}_{xa}}$ (resp. $\|\cdot\|_{W^{m,p}_x}$), with $1 \leqq p \leqq \infty$, $m \in \mathbb{N}$, the Sobolev $W^{m,p}$-norm with respect to $x \in \mathbb{R}/\mathbb{Z}$, $a \in [0, 1]$ (resp. $x \in \mathbb{R}/\mathbb{Z}$). Moreover, we use the notation $\|\cdot\|_{H^m_{xa}}$ for $\|\cdot\|_{W^{m,2}_{xa}}$. We finally note $\|\cdot\|_{\mathscr{C}^{m,\mu}_{xa}}$ (resp. $\|\cdot\|_{\mathscr{C}^{m,\mu}_x}$), with $0 < \mu < 1$, $m \in \mathbb{N}$, the Hölder $\mathscr{C}^{m,\mu}$-norm with respect to $x \in \mathbb{R}/\mathbb{Z}$, $a \in [0, 1]$ (resp. $x \in \mathbb{R}/\mathbb{Z}$).

4.2. Change of variables

Let us suppose that the waterbag continuum $v^\pm(t)$ belongs to H^3_{xa} for t fixed. Since we have the embedding $H^3(\mathbb{R}^2) \hookrightarrow \mathscr{C}^{1,\lambda}(\mathbb{R}^2)$, with $0 < \lambda < 1$, and since $\partial_a v^+ < 0$, $\partial_a v^- > 0$, then using Theorem 1.2 of chapter 2 in [47] the following application

$$a \longmapsto \tau = \begin{cases} v^+(a) & \text{if } \tau \in \Gamma^+ \\ v^-(a) & \text{if } \tau \in \Gamma^- \end{cases} \tag{61}$$

maps [0, 1] one-to-one onto $\Gamma^\pm$ and defines a change of variables for the Cauchy integral. Therefore we have

$$\int_{\Gamma^\pm} \frac{\varphi(\tau)}{\tau - z} \mathrm{d}\tau = \mp \int_0^1 \frac{\varphi\left(v^\pm(\nu)\right)\partial_\nu v^\pm}{v^\pm(\nu) - v^\pm(a)} \mathrm{d}\nu,$$

for all $\varphi \in \mathscr{C}^{0,\lambda}_{xa}$, with $0 < \lambda \leqq 1$. Let us now introduce a generic unknown $X^\pm(t, x, a) \in \{\omega^\pm, \gamma^\pm = 1/\omega^\pm, v^\pm, \mathcal{R}^\pm, r^\pm, k^\pm\}$, $x \in \mathbb{R}/\mathbb{Z}$, $a \in [0, 1]$. Using the change of variables (61) we can write

$$X(\tau) = X(t, x, v^{-1}(\tau)) = \begin{cases} X^+(t, x, (v^+)^{-1}(\tau)) = X^+(t, x, a) & \text{if } \tau \in \Gamma^+ \\ X^-(t, x, (v^-)^{-1}(\tau)) = X^-(t, x, a) & \text{if } \tau \in \Gamma^- \end{cases}$$

Therefore, using the change of variables (61), Riemann invariants (48) and (49) can be rewritten as

$$\mathcal{R}(z) = \mathcal{R}(t, x, v^{-1}(z)) = z + \int_\Gamma \gamma(\tau) \ln|\tau - z| \mathrm{d}\tau \tag{62}$$

$$r = r(t, x) = k + \int_\Gamma \gamma(\tau) \ln|\tau - k| \mathrm{d}\tau \tag{63}$$

Moreover, by differentiating equation (10) with respect to the variable a and by setting $\omega^\pm = \partial_a v^\pm$, $\gamma^\pm = 1/\omega^\pm$ we get the two following equations

$$\partial_t \omega^\pm + v^\pm \partial_x \omega^\pm + \omega^\pm \partial_x v^\pm = 0, \tag{64}$$

$$\partial_t \gamma^\pm + v^\pm \partial_x \gamma^\pm - \gamma^\pm \partial_x v^\pm = 0. \tag{65}$$

4.3. *Idea of the proof*

In order to prove the existence and uniqueness of a classical solution for the waterbag continuum equations (10)–(11) or (13)–(14), we use the quasilinear transport equations (50)–(51) and (64)–(65). The idea is to get a priori energy estimates for the Riemann invariants $\mathcal{R}^{\pm}$, $r^{\pm}$ and the quantities $\omega^{\pm}$ and $\gamma^{\pm}$. To this purpose we must be able to estimate the velocity fields $v^{\pm}$ and $k^{\pm}$ with respect to the quantities $\mathcal{R}^{\pm}$,$r^{\pm}$, $\omega^{\pm}$ and $\gamma^{\pm}$. This latter task is achieved by using the properties of the eigenvalue problem in Section 3.2, the resolution of the Riemann–Hilbert boundary value problem and some classical results in singular integral operator theory and harmonic analysis. The main theorem of this paper, namely Theorem 1, can be found at the end of the paper (Section 4.5).

4.4. *A priori estimates*

In the sequel, the notation $A \lesssim B$ means that there exists a purely numerical constant $\mathcal{C}$ such that $A \leqq \mathcal{C}B$.

4.4.1. A priori estimates for the Riemann invariant $r^{\pm}$ From equation (51) and its first derivatives with respect to x, using integration by part we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|r^{\pm}\|^2_{L^2_x} \lesssim \|k^{\pm}_x\|_{L^\infty_x}\|r^{\pm}\|^2_{L^2_x},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\|r^{\pm}_x\|^2_{L^2_x} \lesssim \|k^{\pm}_x\|_{L^\infty_x}\|r^{\pm}_x\|^2_{L^2_x}.$$

4.4.2. A priori estimates for the Riemann invariant $\mathcal{R}^{\pm}$ From equation (50) and its first derivatives with respect to x, using integration by part and the Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{R}^{\pm}\|^2_{L^2_{xa}} \lesssim \|v^{\pm}_x\|_{L^\infty_{xa}}\|\mathcal{R}^{\pm}\|^2_{L^2_{xa}} \lesssim \|v^{\pm}_x\|_{H^2_{xa}}\|\mathcal{R}^{\pm}\|^2_{L^2_{xa}},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{R}^{\pm}_x\|^2_{L^2_{xa}} \lesssim \|v^{\pm}_x\|_{L^\infty_{xa}}\|\mathcal{R}^{\pm}_x\|^2_{L^2_{xa}} \lesssim \|v^{\pm}_x\|_{H^2_{xa}}\|\mathcal{R}^{\pm}_x\|^2_{L^2_{xa}}.$$

From derivatives of equation (50) with respect to x and a, using integration by part, Young's inequality and the Sobolev embeddings $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, $H^{1/2}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{R}^{\pm}_{xx}\|^2_{L^2_{xa}} &\lesssim \|v^{\pm}_{xxx}\|_{L^2_{xa}}\|\mathcal{R}^{\pm}_x\|^2_{L^4_{xa}} + \|v^{\pm}_x\|_{L^\infty_{xa}}\|\mathcal{R}^{\pm}_{xx}\|^2_{L^2_{xa}},\\ &\lesssim \|v^{\pm}_{xxx}\|_{L^2_{xa}}\|\mathcal{R}^{\pm}_x\|^2_{H^1_{xa}} + \|v^{\pm}_x\|_{H^2_{xa}}\|\mathcal{R}^{\pm}_{xx}\|^2_{L^2_{xa}},\end{aligned}$$
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{R}^{\pm}_{xxx}\|^2_{L^2_{xa}} \lesssim{}& \|v^{\pm}_x\|_{L^\infty_{xa}}\|\mathcal{R}^{\pm}_{xxx}\|^2_{L^2_{xa}} + \|\mathcal{R}^{\pm}_x\|_{L^\infty_{xa}}\|\mathcal{R}^{\pm}_{xxx}\|_{L^2_{xa}}\|v^{\pm}_{xxx}\|_{L^2_{xa}}\\ &+ \|v^{\pm}_{xx}\|_{L^4_{xa}}\|\mathcal{R}^{\pm}_{xx}\|_{L^4_{xa}}\|\mathcal{R}^{\pm}_{xxx}\|_{L^2_{xa}}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|v_x^\pm\|_{H^2_{xa}}\|\mathcal{R}^\pm_{xxx}\|^2_{L^2_{xa}} + \|\mathcal{R}^\pm_x\|_{H^2_{xa}}\|\mathcal{R}^\pm_{xxx}\|_{L^2_{xa}}\|v^\pm_{xxx}\|_{L^2_{xa}} \\
&\quad + \|v^\pm_{xx}\|_{H^1_{xa}}\|\mathcal{R}^\pm_{xx}\|_{H^1_{xa}}\|\mathcal{R}^\pm_{xxx}\|_{L^2_{xa}}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{R}^\pm_{xa}\|^2_{L^2_{xa}} &\lesssim \|\mathcal{R}^\pm_{xa}\|_{L^2_{xa}}\|\omega^\pm_x\|_{L^4_{xa}}\|\mathcal{R}^\pm_x\|_{L^4_{xa}} + \|\omega^\pm\|_{L^\infty_{xa}}\|\mathcal{R}^\pm_{xa}\|_{L^2_{xa}}\|\mathcal{R}^\pm_{xx}\|_{L^2_{xa}} \\
&\quad + \|v^\pm_x\|_{L^\infty_{xa}}\|\mathcal{R}^\pm_{xa}\|^2_{L^2_{xa}} \\
&\lesssim \|\mathcal{R}^\pm_{xa}\|_{L^2_{xa}}\|\omega^\pm_x\|_{H^1_{xa}}\|\mathcal{R}^\pm_x\|_{H^1_{xa}} + \|\omega^\pm\|_{H^2_{xa}}\|\mathcal{R}^\pm_{xa}\|_{L^2_{xa}}\|\mathcal{R}^\pm_{xx}\|_{L^2_{xa}} \\
&\quad + \|v^\pm_x\|_{H^2_{xa}}\|\mathcal{R}^\pm_{xa}\|^2_{L^2_{xa}}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{R}^\pm_{xaa}\|^2_{L^2_{xa}} &\lesssim \|\mathcal{R}^\pm_{xaa}\|_{L^2_{xa}}\|\omega^\pm_{ax}\|_{L^4_{xa}}\|\mathcal{R}^\pm_x\|_{L^4_{xa}} + \|\mathcal{R}^\pm_{xaa}\|_{L^2_{xa}}\|\omega^\pm_a\|_{L^4_{xa}}\|\mathcal{R}^\pm_{xx}\|_{L^4_{xa}} \\
&\quad + \|\mathcal{R}^\pm_{xaa}\|_{L^2_{xa}}\|\omega^\pm_x\|_{L^4_{xa}}\|\mathcal{R}^\pm_{xa}\|_{L^4_{xa}} + \|\mathcal{R}^\pm_{xxa}\|_{L^2_{xa}}\|\omega^\pm\|_{L^\infty_{xa}}\|\mathcal{R}^\pm_{xaa}\|_{L^2_{xa}} \\
&\quad + \|v^\pm_x\|_{L^\infty_{xa}}\|\mathcal{R}^\pm_{xaa}\|^2_{L^2_{xa}} \\
&\lesssim \|\mathcal{R}^\pm_{xaa}\|_{L^2_{xa}}\|\omega^\pm_{ax}\|_{H^1_{xa}}\|\mathcal{R}^\pm_x\|_{H^1_{xa}} + \|\mathcal{R}^\pm_{xaa}\|_{L^2_{xa}}\|\omega^\pm_a\|_{H^1_{xa}}\|\mathcal{R}^\pm_{xx}\|_{H^1_{xa}} \\
&\quad + \|\mathcal{R}^\pm_{xaa}\|_{L^2_{xa}}\|\omega^\pm_x\|_{H^1_{xa}}\|\mathcal{R}^\pm_{xa}\|_{H^1_{xa}} + \|\mathcal{R}^\pm_{xxa}\|_{L^2_{xa}}\|\omega^\pm\|_{H^2_{xa}}\|\mathcal{R}^\pm_{xaa}\|_{L^2_{xa}} \\
&\quad + \|v^\pm_x\|_{H^2_{xa}}\|\mathcal{R}^\pm_{xaa}\|^2_{L^2_{xa}}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{R}^\pm_{xxa}\|^2_{L^2_{xa}} &\lesssim \|\mathcal{R}^\pm_{xxa}\|_{L^2_{xa}}\|\omega^\pm_{xx}\|_{L^4_{xa}}\|\mathcal{R}^\pm_x\|_{L^4_{xa}} + \|\mathcal{R}^\pm_{xxa}\|_{L^2_{xa}}\|v^\pm_{xx}\|_{L^4_{xa}}\|\mathcal{R}^\pm_{xa}\|_{L^4_{xa}} \\
&\quad + \|\mathcal{R}^\pm_{xxa}\|_{L^2_{xa}}\|\omega^\pm_x\|_{L^4_{xa}}\|\mathcal{R}^\pm_{xx}\|_{L^4_{xa}} + \|v^\pm_x\|_{L^\infty_{xa}}\|\mathcal{R}^\pm_{xxa}\|^2_{L^2_{xa}} \\
&\quad + \|\mathcal{R}^\pm_{xxa}\|_{L^2_{xa}}\|\omega^\pm\|_{L^\infty_{xa}}\|\mathcal{R}^\pm_{xxx}\|_{L^2_{xa}} \\
&\lesssim \|\mathcal{R}^\pm_{xxa}\|_{L^2_{xa}}\|\omega^\pm_{xx}\|_{H^1_{xa}}\|\mathcal{R}^\pm_x\|_{H^1_{xa}} + \|\mathcal{R}^\pm_{xxa}\|_{L^2_{xa}}\|v^\pm_{xx}\|_{H^1_{xa}}\|\mathcal{R}^\pm_{xa}\|_{H^1_{xa}} \\
&\quad + \|\mathcal{R}^\pm_{xxa}\|_{L^2_{xa}}\|\omega^\pm_x\|_{H^1_{xa}}\|\mathcal{R}^\pm_{xx}\|_{H^1_{xa}} + \|v^\pm_x\|_{H^2_{xa}}\|\mathcal{R}^\pm_{xxa}\|^2_{L^2_{xa}} \\
&\quad + \|\mathcal{R}^\pm_{xxa}\|_{L^2_{xa}}\|\omega^\pm\|_{H^2_{xa}}\|\mathcal{R}^\pm_{xxx}\|_{L^2_{xa}}.
\end{aligned}
$$

4.4.3. A priori estimates for $\omega^\pm$ By differentiating equation (64) with respect to x and a as often as it is needed, using integration by part, Young's inequality and the Sobolev embeddings $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, $H^{1/2}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$, $H^{2/3}(\mathbb{R}^2) \hookrightarrow L^6(\mathbb{R}^2)$ we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^\pm\|^2_{L^2_{xa}} &\lesssim \|v^\pm_x\|_{L^\infty_{xa}}\|\omega^\pm\|^2_{L^2_{xa}} \lesssim \|v^\pm_x\|_{H^2_{xa}}\|\omega^\pm\|^2_{L^2_{xa}}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^\pm_x\|^2_{L^2_{xa}} &\lesssim \|v^\pm_x\|_{L^\infty_{xa}}\|\omega^\pm_x\|^2_{L^2_{xa}} + \|v^\pm_{xx}\|_{L^2_{xa}}\|\omega^\pm_x\|_{L^2_{xa}}\|\omega^\pm\|_{L^\infty_{xa}} \\
&\lesssim \|v^\pm_x\|_{H^2_{xa}}\|\omega^\pm_x\|^2_{L^2_{xa}} + \|v^\pm_{xx}\|_{L^2_{xa}}\|\omega^\pm_x\|_{L^2_{xa}}\|\omega^\pm\|_{H^2_{xa}}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^\pm_a\|^2_{L^2_{xa}} &\lesssim \|\omega^\pm\|_{L^\infty_{xa}}\|\omega^\pm_x\|_{L^2_{xa}}\|\omega^\pm_a\|_{L^2_{xa}} + \|v^\pm_x\|_{L^\infty_{xa}}\|\omega^\pm_a\|^2_{L^2_{xa}} \\
&\lesssim \|\omega^\pm\|_{H^2_{xa}}\|\omega^\pm_x\|_{L^2_{xa}}\|\omega^\pm_a\|_{L^2_{xa}} + \|v^\pm_x\|_{H^2_{xa}}\|\omega^\pm_a\|^2_{L^2_{xa}},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^{\pm}_{xx}\|^2_{L^2_{xa}} &\lesssim \|\omega^{\pm}\|_{L^\infty_{xa}}\|\omega^{\pm}_{xx}\|_{L^2_{xa}}\|v^{\pm}_{xxx}\|_{L^2_{xa}} + \|v^{\pm}_{xxx}\|_{L^2_{xa}}\|\omega^{\pm}_{x}\|^2_{L^4_{xa}} \\
&\quad + \|v^{\pm}_{x}\|_{L^\infty_{xa}}\|\omega^{\pm}_{xx}\|^2_{L^2_{xa}} \lesssim \|\omega^{\pm}\|_{H^2_{xa}}\|\omega^{\pm}_{xx}\|_{L^2_{xa}}\|v^{\pm}_{xxx}\|_{L^2_{xa}} \\
&\quad + \|v^{\pm}_{xxx}\|_{L^2_{xa}}\|\omega^{\pm}_{x}\|^2_{H^1_{xa}} + \|v^{\pm}_{x}\|_{H^2_{xa}}\|\omega^{\pm}_{xx}\|^2_{L^2_{xa}}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^{\pm}_{aa}\|^2_{L^2_{xa}} &\lesssim \|\omega^{\pm}\|_{L^\infty_{xa}}\|\omega^{\pm}_{aa}\|_{L^2_{xa}}\|\omega^{\pm}_{xa}\|_{L^2_{xa}} + \|\omega^{\pm}_{aa}\|_{L^2_{xa}}\|\omega^{\pm}_{x}\|_{L^4_{xa}}\|\omega^{\pm}_{a}\|_{L^4_{xa}} \\
&\quad + \|v^{\pm}_{x}\|_{L^\infty_{xa}}\|\omega^{\pm}_{aa}\|^2_{L^2_{xa}} \\
&\lesssim \|\omega^{\pm}\|_{H^2_{xa}}\|\omega^{\pm}_{aa}\|_{L^2_{xa}}\|\omega^{\pm}_{ax}\|_{L^2_{xa}} + \|\omega^{\pm}_{aa}\|_{L^2_{xa}}\|\omega^{\pm}_{x}\|_{H^1_{xa}}\|\omega^{\pm}_{a}\|_{H^1_{xa}} \\
&\quad + \|v^{\pm}_{x}\|_{H^2_{xa}}\|\omega^{\pm}_{aa}\|^2_{L^2_{xa}}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^{\pm}_{xa}\|^2_{L^2_{xa}} &\lesssim \|\omega^{\pm}\|_{L^\infty_{xa}}\|\omega^{\pm}_{x}\|_{L^2_{xa}}\|\omega^{\pm}_{xa}\|_{L^2_{xa}} + \|v^{\pm}_{x}\|_{L^\infty_{xa}}\|\omega^{\pm}_{xa}\|_{L^2_{xa}}\|\omega^{\pm}_{a}\|_{L^2_{xa}} \\
&\quad + \|v^{\pm}_{x}\|_{L^\infty_{xa}}\|\omega^{\pm}_{xa}\|^2_{L^2_{xa}} \\
&\lesssim \|\omega^{\pm}\|_{H^2_{xa}}\|\omega^{\pm}_{x}\|_{L^2_{xa}}\|\omega^{\pm}_{xa}\|_{L^2_{xa}} + \|v^{\pm}_{x}\|_{H^2_{xa}}\|\omega^{\pm}_{xa}\|_{L^2_{xa}}\|\omega^{\pm}_{a}\|_{L^2_{xa}} \\
&\quad + \|v^{\pm}_{x}\|_{H^2_{xa}}\|\omega^{\pm}_{xa}\|^2_{L^2_{xa}}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^{\pm}_{xxx}\|^2_{L^2_{xa}} &\lesssim \|v^{\pm}_{x}\|_{L^\infty_{xa}}\|\omega^{\pm}_{xxx}\|^2_{L^2_{xa}} + \|\omega^{\pm}\|_{L^\infty_{xa}}\|\omega^{\pm}_{xxx}\|_{L^2_{xa}}\|v^{\pm}_{xxx}\|_{L^2_{xa}} \\
&\quad + \|v^{\pm}_{xxx}\|_{L^2_{xa}}\|\omega^{\pm}_{xx}\|^2_{L^4_{xa}} \\
&\lesssim \|v^{\pm}_{x}\|_{H^2_{xa}}\|\omega^{\pm}_{xxx}\|^2_{L^2_{xa}} + \|\omega^{\pm}\|_{H^2_{xa}}\|\omega^{\pm}_{xxx}\|_{L^2_{xa}}\|v^{\pm}_{xxx}\|_{L^2_{xa}} \\
&\quad + \|v^{\pm}_{xxx}\|_{L^2_{xa}}\|\omega^{\pm}_{xx}\|^2_{H^1_{xa}}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^{\pm}_{xxa}\|^2_{L^2_{xa}} &\lesssim \|v^{\pm}_{x}\|_{L^\infty_{xa}}\|\omega^{\pm}_{xxa}\|^2_{L^2_{xa}} + \|v^{\pm}_{xxx}\|_{L^2_{xa}}\|\omega^{\pm}_{xa}\|^2_{L^4_{xa}} \\
&\quad + \|\omega^{\pm}_{xxa}\|_{L^2_{xa}}\|\omega^{\pm}_{x}\|_{L^4_{xa}}\|\omega^{\pm}_{xx}\|_{L^4_{xa}} + \|\omega^{\pm}_{xxa}\|_{L^2_{xa}}\|\omega^{\pm}_{x}\|^3_{L^6_{xa}} \\
&\quad + \|\omega^{\pm}\|_{L^\infty_{xa}}\|\omega^{\pm}_{xxa}\|_{L^2_{xa}}\|\omega^{\pm}_{x}\|_{L^4_{xa}}\|\omega^{\pm}_{xx}\|_{L^4_{xa}} \\
&\quad + \|\omega^{\pm}_{xxa}\|_{L^2_{xa}}\|v^{\pm}_{xxx}\|_{L^2_{xa}}\|\omega^{\pm}_{a}\|_{L^\infty_{xa}} \\
&\lesssim \|v^{\pm}_{x}\|_{H^2_{xa}}\|\omega^{\pm}_{xxa}\|^2_{L^2_{xa}} + \|v^{\pm}_{xxx}\|_{L^2_{xa}}\|\omega^{\pm}_{xa}\|^2_{H^1_{xa}} \\
&\quad + \|\omega^{\pm}_{xxa}\|_{L^2_{xa}}\|\omega^{\pm}_{x}\|_{H^1_{xa}}\|\omega^{\pm}_{xx}\|_{H^1_{xa}} + \|\omega^{\pm}_{xxa}\|_{L^2_{xa}}\|\omega^{\pm}_{x}\|^3_{H^1_{xa}} \\
&\quad + \|\omega^{\pm}\|_{H^2_{xa}}\|\omega^{\pm}_{xxa}\|_{L^2_{xa}}\|\omega^{\pm}_{x}\|_{H^1_{xa}}\|\omega^{\pm}_{xx}\|_{H^1_{xa}} \\
&\quad + \|\omega^{\pm}_{xxa}\|_{L^2_{xa}}\|v^{\pm}_{xxx}\|_{L^2_{xa}}\|\omega^{\pm}_{a}\|_{H^2_{xa}}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^{\pm}_{xaa}\|^2_{L^2_{xa}} &\lesssim \|\omega^{\pm}\|_{L^\infty_{xa}}\|\omega^{\pm}_{xaa}\|_{L^2_{xa}}\|\omega^{\pm}_{xxa}\|_{L^2_{xa}} + \|\omega^{\pm}_{xaa}\|_{L^2_{xa}}\|\omega^{\pm}_{xx}\|_{L^4_{xa}}\|\omega^{\pm}_{a}\|_{L^4_{xa}} \\
&\quad + \|\omega^{\pm}_{xaa}\|_{L^2_{xa}}\|v^{\pm}_{xx}\|_{L^4_{xa}}\|\omega^{\pm}_{aa}\|_{L^4_{xa}} + \|\omega^{\pm}_{xaa}\|_{L^2_{xa}}\|\omega^{\pm}_{xa}\|_{L^4_{xa}}\|\omega^{\pm}_{x}\|_{L^4_{xa}} \\
&\quad + \|\omega^{\pm}_{xaa}\|_{L^2_{xa}}\|\omega^{\pm}_{xa}\|^2_{L^4_{xa}} + \|v^{\pm}_{x}\|_{L^\infty_{xa}}\|\omega^{\pm}_{xaa}\|^2_{L^2_{xa}} \\
&\lesssim \|\omega^{\pm}\|_{H^2_{xa}}\|\omega^{\pm}_{xaa}\|_{L^2_{xa}}\|\omega^{\pm}_{xxa}\|_{L^2_{xa}} + \|\omega^{\pm}_{xaa}\|_{L^2_{xa}}\|\omega^{\pm}_{xx}\|_{H^1_{xa}}\|\omega^{\pm}_{a}\|_{H^1_{xa}} \\
&\quad + \|\omega^{\pm}_{xaa}\|_{L^2_{xa}}\|v^{\pm}_{xx}\|_{H^1_{xa}}\|\omega^{\pm}_{aa}\|_{H^1_{xa}} + \|\omega^{\pm}_{xaa}\|_{L^2_{xa}}\|\omega^{\pm}_{xa}\|_{H^1_{xa}}\|\omega^{\pm}_{x}\|_{H^1_{xa}}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\omega^\pm_{xaa}\|_{L^2_{xa}}\|\omega^\pm_{xa}\|^2_{H^1_{xa}}\|v^\pm_x\|_{H^2_{xa}}\|\omega^\pm_{xaa}\|^2_{L^2_{xa}},\\
\frac{\mathrm{d}}{\mathrm{d}t}\|\omega^\pm_{aaa}\|^2_{L^2_{xa}} \lesssim{}& \|\omega^\pm\|_{L^\infty_{xa}}\|\omega^\pm_{xaa}\|_{L^2_{xa}}\|\omega^\pm_{aaa}\|_{L^2_{xa}}+\|\omega^\pm_{aaa}\|_{L^2_{xa}}\|\omega^\pm_{aa}\|_{L^4_{xa}}\|\omega^\pm_x\|_{L^4_{xa}}\\
&+\|\omega^\pm\|_{L^\infty_{xa}}\|\omega^\pm_{aaa}\|_{L^2_{xa}}\|\omega^\pm_{xa}\|_{L^2_{xa}}+\|\omega^\pm_{aaa}\|_{L^2_{xa}}\|\omega^\pm_{a}\|_{L^4_{xa}}\|\omega^\pm_x\|_{L^4_{xa}}\\
&\|v^\pm_x\|_{L^\infty_{xa}}\|\omega^\pm_{aaa}\|^2_{L^2_{xa}}\\
\lesssim{}& \|\omega^\pm\|_{H^2_{xa}}\|\omega^\pm_{xaa}\|_{L^2_{xa}}\|\omega^\pm_{aaa}\|_{L^2_{xa}}+\|\omega^\pm_{aaa}\|_{L^2_{xa}}\|\omega^\pm_{aa}\|_{H^1_{xa}}\|\omega^\pm_x\|_{H^1_{xa}}\\
&+\|\omega^\pm\|_{H^2_{xa}}\|\omega^\pm_{aaa}\|_{L^2_{xa}}\|\omega^\pm_{xa}\|_{L^2_{xa}}+\|\omega^\pm_{aaa}\|_{L^2_{xa}}\|\omega^\pm_{a}\|_{H^1_{xa}}\|\omega^\pm_x\|_{H^1_{xa}}\\
&\|v^\pm_x\|_{H^2_{xa}}\|\omega^\pm_{aaa}\|^2_{L^2_{xa}}.
\end{aligned}
$$

4.4.4. A priori estimates for $\gamma^\pm$ As the structure of the equation (65) is the same as the structure of the equation (64), a priori estimates for $\gamma^\pm$ are the same as for $\omega^\pm$ and are obtained straightforwardly by substituting $\gamma^\pm$ for $\omega^\pm$.

4.4.5. A priori estimates for $v^\pm$ Sometimes, to simplify the notation, we omit the dependence of functions on the variables t and x.

By differentiating the equation (48) three times with respect to x, we obtain

$$
\begin{aligned}
\mathcal{R}^\pm_x(t,x,a) = A(v^\pm(a))v^\pm_x(a) + \int_0^1\Bigg(&\frac{v^-_x(\nu)}{v^-(\nu)-v^\pm(a)}\\
&-\frac{v^+_x(\nu)}{v^+(\nu)-v^\pm(a)}\Bigg)\mathrm{d}\nu,
\end{aligned}\tag{66}
$$

$$
\begin{aligned}
&\mathcal{R}^\pm_{xx}(t,x,a) + \int_0^1\left\{\left(\frac{v^-_x(\nu)-v^\pm_x(a)}{v^-(\nu)-v^\pm(a)}\right)^2-\left(\frac{v^+_x(\nu)-v^\pm_x(a)}{v^+(\nu)-v^\pm(a)}\right)^2\right\}\mathrm{d}\nu\\
&= A(v^\pm(a))v^\pm_{xx}(a) + \int_0^1\left(\frac{v^-_{xx}(\nu)}{v^-(\nu)-v^\pm(a)}-\frac{v^+_{xx}(\nu)}{v^+(\nu)-v^\pm(a)}\right)\mathrm{d}\nu,
\end{aligned}\tag{67}
$$

and

$$
\begin{aligned}
&\mathcal{R}^\pm_{xxx}(t,x,a) - 2\int_0^1\left\{\left(\frac{v^-_x(\nu)-v^\pm_x(a)}{v^-(\nu)-v^\pm(a)}\right)^3-\left(\frac{v^+_x(\nu)-v^\pm_x(a)}{v^+(\nu)-v^\pm(a)}\right)^3\right\}\mathrm{d}\nu\\
&\quad+3\int_0^1\Bigg\{\left(\frac{v^-_{xx}(\nu)-v^\pm_{xx}(a)}{v^-(\nu)-v^\pm(a)}\right)\left(\frac{v^-_x(\nu)-v^\pm_x(a)}{v^-(\nu)-v^\pm(a)}\right)\\
&\quad-\left(\frac{v^+_{xx}(\nu)-v^\pm_{xx}(a)}{v^+(\nu)-v^\pm(a)}\right)\left(\frac{v^+_x(\nu)-v^\pm_x(a)}{v^+(\nu)-v^\pm(a)}\right)\Bigg\}\mathrm{d}\nu\\
&= A(v^\pm(a))v^\pm_{xxx}(a) + \int_0^1\left(\frac{v^-_{xxx}(\nu)}{v^-(\nu)-v^\pm(a)}-\frac{v^+_{xxx}(\nu)}{v^+(\nu)-v^\pm(a)}\right)\mathrm{d}\nu.
\end{aligned}\tag{68}
$$

Using the change of variables (61), equations (66), (67) and (68) can be recast respectively as

$$\mathcal{R}_x(z) = A(z)v_x(z) + \int_\Gamma \frac{v_x(\tau)}{\tau - z}\frac{\mathrm{d}\tau}{\omega(\tau)} = (\mathscr{P}_+\chi^-(z) + \mathscr{P}_-\chi^+(z))v_x(z), \tag{69}$$

$$\mathcal{R}_{xx}(z) + \int_\Gamma \left(\frac{v_x(\tau) - v_x(z)}{\tau - z}\right)^2 \frac{\mathrm{d}\tau}{\omega(\tau)} = A(z)v_{xx}(z) + \int_\Gamma \frac{v_{xx}(\tau)}{\tau - z}\frac{\mathrm{d}\tau}{\omega(\tau)} = (\mathscr{P}_+\chi^-(z) + \mathscr{P}_-\chi^+(z))v_{xx}(z), \tag{70}$$

and

$$\mathcal{R}_{xxx}(z) - 2\int_\Gamma \left(\frac{v_x(\tau) - v_x(z)}{\tau - z}\right)^3 \frac{\mathrm{d}\tau}{\omega(\tau)} + 3\int_\Gamma \left(\frac{v_{xx}(\tau) - v_{xx}(z)}{\tau - z}\right) \times \left(\frac{v_x(\tau) - v_x(z)}{\tau - z}\right)\frac{\mathrm{d}\tau}{\omega(\tau)} = A(z)v_{xxx}(z) + \int_\Gamma \frac{v_{xxx}(\tau)}{\tau - z}\frac{\mathrm{d}\tau}{\omega(\tau)} = (\mathscr{P}_+\chi^-(z) + \mathscr{P}_-\chi^+(z))v_{xxx}(z). \tag{71}$$

In equations (66)–(71) the integrals must be evaluated in the sense of the principal value. For arbitrary piecewise smooth curve Γ, we denote by $\mathbb{PX}(\Gamma, \mathcal{D}_\Gamma)$ the space of piecewise smooth functions with jump discontinuity at $\tau \in \mathcal{D}_\Gamma = \{v_c\}$: $\mathbb{PX}(\Gamma, \mathcal{D}_\Gamma) := \left\{g \in \mathbb{X}(\Gamma^\pm)\right\}$, where $\mathbb{X}(\Gamma^\pm)$ denotes one of the spaces $\mathscr{C}^m(\Gamma^\pm)$, $\mathscr{C}^{m,\mu}(\Gamma^\pm)$, with $0 < \mu < 1$. Since we have supposed that for t fixed, $v^\pm(t) \in H^3_{xa}$ and since we have the embedding $H^3(\mathbb{R}^2) \hookrightarrow \mathscr{C}^{1,\mu}(\mathbb{R}^2)$, with $0 < \mu < 1$, then $v^\pm(t) \in \mathscr{C}^{1,\mu}_{xa}$ and therefore $v(t) \in \mathbb{P}\mathscr{C}^{1,\mu}(\{\mathbb{R}/\mathbb{Z}\} \times \Gamma, \mathcal{D}_\Gamma)$, $\omega(t) \in \mathbb{P}\mathscr{C}^{0,\mu}(\{\mathbb{R}/\mathbb{Z}\} \times \Gamma, \mathcal{D}_\Gamma)$ and $\gamma(t) \in \mathbb{P}\mathscr{C}^{0,\mu}(\{\mathbb{R}/\mathbb{Z}\} \times \Gamma, \mathcal{D}_\Gamma)$. Since $\omega(t) \in \mathbb{P}\mathscr{C}^{0,\mu}(\{\mathbb{R}/\mathbb{Z}\} \times \Gamma, \mathcal{D}_\Gamma)$ and $\gamma(t) \in \mathbb{P}\mathscr{C}^{0,\mu}(\{\mathbb{R}/\mathbb{Z}\} \times \Gamma, \mathcal{D}_\Gamma)$, under the condition of Lemma 1, the operator $\mathscr{P}_+\chi^-\mathcal{I} + \mathscr{P}_-\chi^+\mathcal{I}$ is a Fredholm right-invertible operator on the space $L^2(\Gamma)$. Let us now compute the inverse of the operator $\mathscr{P}_+\chi^-\mathcal{I} + \mathscr{P}_-\chi^+\mathcal{I}$, which is equivalent to solving a non-homogeneous Riemann–Hilbert boundary value problem [30,33,34,52,54]. Let us first invert the equation (69). Let us set

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{B(\tau)v_x(\tau)}{\tau - z}\mathrm{d}\tau, \quad z \in \mathbb{C} \setminus \Gamma,$$

where $B(\tau) = i\pi/\omega(\tau) = i\pi/\omega(t, x, v^{-1}(\tau)) = i\pi\gamma(\tau) = i\pi\gamma(t, x, v^{-1}(\tau)) \neq 0$ and $\Phi(\infty) = 0$. Using the Sokhotskii–Plemelj formula [30,52,54], the limiting values of the sectionally analytic function $\Phi(z)$ from the upper and lower half-planes in the section Γ are given by

$$\Phi^\pm(z) = \pm\frac{1}{2}B(z)v_x(z) + \frac{1}{2\pi i}\text{p.v.}\int_\Gamma \frac{B(\tau)v_x(\tau)}{\tau - z}\mathrm{d}\tau = \pm\mathscr{P}_\pm B(z)v_x(z), \quad z \in \Gamma.$$

By observing that

$$\Phi^+(z) + \Phi^-(z) = \frac{1}{\pi}\text{p.v.}\int_\Gamma \frac{B(\tau)v_x(\tau)}{\tau - z}\mathrm{d}\tau, \quad \text{and} \quad \Phi^+(z) - \Phi^-(z) = B(z)v_x(z),$$

then equation (69) is equivalent to

$$\frac{A(z)}{B(z)}\left(\Phi^{+}(z)-\Phi^{-}(z)\right)+\Phi^{+}(z)+\Phi^{-}(z)=\mathcal{R}_x(z)$$

or

$$\Phi^{+}(z)=\frac{A(z)-B(z)}{A(z)+B(z)}\Phi^{-}(z)+\frac{B(z)\mathcal{R}_x(z)}{A(z)+B(z)},\quad z\in\Gamma. \tag{72}$$

The solution of the homogeneous Riemann–Hilbert boundary value problem

$$\Phi^{+}(z)=\frac{A(z)-B(z)}{A(z)+B(z)}\Phi^{-}(z),\quad z\in\Gamma, \tag{73}$$

have been computed in Section 3.3.2 and is equal to $\mathcal{X}P_1=\chi P_1/\varrho$, where $P_1(z)=\alpha_0+\alpha_1 z$ is a polynomial of degree one. We now look for the general solution of the non-homogeneous Riemann–Hilbert boundary value problem (72) which is the sum of the general solution of the homogeneous Riemann–Hilbert boundary value problem (73) and a particular solution of the non-homogeneous one [30,33,34,52,54]. Since the function $\mathcal{X}$ is the canonical solution of the problem (73) (see Section 3.3.2), the problem (72) is equivalent to finding the function $\Psi(z)$, sectionally holomorphic in $\mathbb{C}\setminus\Gamma$, whose boundary values Ψ^{+} and Ψ^{-} satisfy the boundary condition

$$\Psi^{+}(z)-\Psi^{-}(z)=\frac{\varrho(z)B(z)\mathcal{R}_x(z)}{|\chi^{+}(z)|^2},\quad z\in\Gamma, \tag{74}$$

where $\Psi=\Phi/\mathcal{X}$. Using the Sokhotskii–Plemelj formula [30,52,54], we see that

$$\Psi(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\varrho(\tau)B(\tau)\mathcal{R}_x(\tau)}{|\chi^{+}(\tau)|^2(\tau-z)}\mathrm{d}\tau,\quad z\in\mathbb{C}\setminus\Gamma, \tag{75}$$

is a solution of (74), with $\Psi(\infty)=0$. From (72)–(75) we have

$$\left[\frac{\Phi(z)}{\mathcal{X}(z)}-\Psi(z)\right]^{+}=\left[\frac{\Phi(z)}{\mathcal{X}(z)}-\Psi(z)\right]^{-},\quad z\in\Gamma.$$

The last relation indicates that the function $[\Phi/\mathcal{X}-\Psi]^{+}$ analytic in the upper plane and the function $[\Phi/\mathcal{X}-\Psi]^{-}$ analytic in the lower plane constitute the analytic continuation of each other through the contour Γ. According to the generalized Liouville Theorem [1,30], the holomorphic function $\Phi/\mathcal{X}-\Psi$ must reduce to a polynomial of second degree, that is, $\Phi=\mathcal{X}(\Psi+P_2)$. Since $\Psi(\infty)=\Phi(\infty)=0$ and $\chi(\infty)=1$, the coefficient of the monomial of highest degree in P_2 is zero, hence $P_2(z)$ reduces to $P_1(z)=\alpha_0+\alpha_1 z$, a polynomial of degree one. Finally the solution of (72) is given by

$$\Phi(z)=\frac{\chi(z)}{\varrho(z)}\left(P_1(z)+\frac{1}{2\pi i}\int_{\Gamma}\frac{\varrho(\tau)B(\tau)\mathcal{R}_x(\tau)}{|\chi^{+}(\tau)|^2(\tau-z)}\mathrm{d}\tau\right),\quad z\in\mathbb{C}\setminus\Gamma.$$

Therefore, using the Sokhotskii–Plemelj formula [30,52,54], we obtain

$$v_x(z) = \frac{\Phi^+(z) - \Phi^-(z)}{B(z)} = -2\frac{P_1(z)}{\varrho(z)} + \frac{A(z)\mathcal{R}_x(z)}{|\chi^+(z)|^2} - \frac{1}{\varrho(z)}\int_\Gamma \frac{\varrho(\tau)\gamma(\tau)\mathcal{R}_x(\tau)}{|\chi^+(\tau)|^2(\tau - z)}\mathrm{d}\tau, \quad z \in \Gamma. \quad (76)$$

In the same way we can invert equations (70) and (71) to obtain respectively, for $z \in \Gamma$,

$$\begin{aligned} v_{xx}(z) = &-2\frac{P_1(z)}{\varrho(z)} + \frac{A(z)}{|\chi^+(z)|^2}\left(\mathcal{R}_{xx}(z) + \int_\Gamma \left(\frac{v_x(\tau) - v_x(z)}{\tau - z}\right)^2 \gamma(\tau)\mathrm{d}\tau\right) \\ &- \frac{1}{\varrho(z)}\int_\Gamma \frac{\varrho(\tau)\gamma(\tau)\mathcal{R}_{xx}(\tau)}{|\chi^+(\tau)|^2(\tau - z)}\mathrm{d}\tau \\ &- \frac{1}{\varrho(z)}\int_\Gamma\int_\Gamma \frac{\varrho(\tau_1)\gamma(\tau_1)\gamma(\tau_2)}{|\chi^+(\tau_1)|^2(\tau_1 - z)}\left(\frac{v_x(\tau_2) - v_x(\tau_1)}{\tau_2 - \tau_1}\right)^2 \mathrm{d}\tau_2\mathrm{d}\tau_1, \end{aligned} \quad (77)$$

and

$$\begin{aligned} v_{xxx}(z) = &-2\frac{P_1(z)}{\varrho(z)} + \frac{A(z)}{|\chi^+(z)|^2}\left(\mathcal{R}_{xxx}(z) - 2\int_\Gamma \left(\frac{v_x(\tau) - v_x(z)}{\tau - z}\right)^3 \gamma(\tau)\mathrm{d}\tau\right. \\ &\left. + 3\int_\Gamma \left(\frac{v_{xx}(\tau) - v_{xx}(z)}{\tau - z}\right)\left(\frac{v_x(\tau) - v_x(z)}{\tau - z}\right)\gamma(\tau)\mathrm{d}\tau\right) \\ &- \frac{1}{\varrho(z)}\int_\Gamma \frac{\varrho(\tau)\gamma(\tau)\mathcal{R}_{xxx}(\tau)}{|\chi^+(\tau)|^2(\tau - z)}\mathrm{d}\tau + \frac{2}{\varrho(z)}\int_\Gamma\int_\Gamma \frac{\varrho(\tau_1)\gamma(\tau_1)\gamma(\tau_2)}{|\chi^+(\tau_1)|^2(\tau_1 - z)} \\ &\times \left(\frac{v_x(\tau_2) - v_x(\tau_1)}{\tau_2 - \tau_1}\right)^3 \mathrm{d}\tau_2\mathrm{d}\tau_1 - \frac{3}{\varrho(z)}\int_\Gamma\int_\Gamma \frac{\varrho(\tau_1)\gamma(\tau_1)\gamma(\tau_2)}{|\chi^+(\tau_1)|^2(\tau_1 - z)} \\ &\times \left(\frac{v_{xx}(\tau_2) - v_{xx}(\tau_1)}{\tau_2 - \tau_1}\right)\left(\frac{v_x(\tau_2) - v_x(\tau_1)}{\tau_2 - \tau_1}\right)\mathrm{d}\tau_2\mathrm{d}\tau_1. \end{aligned} \quad (78)$$

Using the change of variables (61) equations (76)–(78) can be recast respectively as

$$\begin{aligned} v_x^\pm(t,x,a) = &-2\frac{P_1(v^\pm(t,x,a))}{\varrho(v^\pm(t,x,a))} + \frac{A(v^\pm(t,x,a))\mathcal{R}_x^\pm(t,x,a)}{|\chi^+(v^\pm(t,x,a))|^2} \\ &- \frac{1}{\varrho(v^\pm(t,x,a))}\int_0^1 \frac{\varrho(v^-(t,x,\nu))|\chi^+(v^-(t,x,\nu))|^{-2}\mathcal{R}_x^-(t,x,\nu)}{(v^-(t,x,\nu) - v^\pm(t,x,a))}\mathrm{d}\nu \\ &+ \frac{1}{\varrho(v^\pm(t,x,a))}\int_0^1 \frac{\varrho(v^+(t,x,\nu))|\chi^+(v^+(t,x,\nu))|^{-2}\mathcal{R}_x^+(t,x,\nu)}{(v^+(t,x,\nu) - v^\pm(t,x,a))}\mathrm{d}\nu, \end{aligned} \quad (79)$$

$$\begin{aligned} v_{xx}^\pm(t,x,a) = &-2\frac{P_1(v^\pm(t,x,a))}{\varrho(v^\pm(t,x,a))} + \frac{A(v^\pm(t,x,a))}{|\chi^+(v^\pm(t,x,a))|^2}\left(\mathcal{R}_{xx}^\pm(t,x,a)\right. \\ &\left. + \int_0^1 \left(\frac{v_x^-(t,x,\nu) - v_x^\pm(t,x,a)}{v^-(t,x,\nu) - v^\pm(t,x,a)}\right)^2 \mathrm{d}\nu - \int_0^1 \left(\frac{v_x^+(t,x,\nu) - v_x^\pm(t,x,a)}{v^+(t,x,\nu) - v^\pm(t,x,a)}\right)^2 \mathrm{d}\nu\right) \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\varrho(v^{\pm}(t,x,a))}\int_0^1 \frac{\varrho(v^-(t,x,\nu))|\chi^+(v^-(t,x,\nu))|^{-2}\mathcal{R}^-_{xx}(t,x,\nu)}{(v^-(t,x,\nu)-v^{\pm}(t,x,a))}\mathrm{d}\nu \\
&+\frac{1}{\varrho(v^{\pm}(t,x,a))}\int_0^1 \frac{\varrho(v^+(t,x,\nu))|\chi^+(v^+(t,x,\nu))|^{-2}\mathcal{R}^+_{xx}(t,x,\nu)}{(v^-(t,x,\nu)-v^{\pm}(t,x,a))}\mathrm{d}\nu \\
&-\frac{1}{\varrho(v^{\pm}(t,x,a))}\int_0^1\int_0^1 \left\{\frac{\varrho(v^-(t,x,\nu))|\chi^+(v^-(t,x,\nu_1))|^{-2}}{(v^-(t,x,\nu_1)-v^{\pm}(t,x,a))}\right. \\
&\left.-\frac{\varrho(v^+(t,x,\nu))|\chi^+(v^+(t,x,\nu_1))|^{-2}}{(v^+(t,x,\nu_1)-v^{\pm}(t,x,a))}\right\} \\
&\times\left\{\left(\frac{v^-_x(t,x,\nu_2)-v^{\pm}_x(t,x,\nu_1)}{v^-(t,x,\nu_2)-v^{\pm}(t,x,\nu_1)}\right)^2-\left(\frac{v^+_x(t,x,\nu_2)-v^{\pm}_x(t,x,\nu_1)}{v^+(t,x,\nu_2)-v^{\pm}(t,x,\nu_1)}\right)^2\right\}\mathrm{d}\nu_2\mathrm{d}\nu_1,
\end{aligned}
\tag{80}
$$

and

$$
\begin{aligned}
v^{\pm}_{xxx}(t,x,a) &= -2\frac{P_1(v^{\pm}(t,x,a))}{\varrho(v^{\pm}(t,x,a))}+\frac{A(v^{\pm}(t,x,a))}{|\chi^+(v^{\pm}(t,x,a))|^2}\Big(\mathcal{R}^{\pm}_{xxx}(t,x,a) \\
&-2\int_0^1\left(\frac{v^-_x(t,x,\nu)-v^{\pm}_x(t,x,a)}{v^-(t,x,\nu)-v^{\pm}(t,x,a)}\right)^3\mathrm{d}\nu \\
&+2\int_0^1\left(\frac{v^+_x(t,x,\nu)-v^{\pm}_x(t,x,a)}{v^+(t,x,\nu)-v^{\pm}(t,x,a)}\right)^3\mathrm{d}\nu \\
&+3\int_0^1\left(\frac{v^-_{xx}(t,x,\nu)-v^{\pm}_{xx}(t,x,a)}{v^-(t,x,\nu)-v^{\pm}(t,x,a)}\right)\left(\frac{v^-_x(t,x,\nu)-v^{\pm}_x(t,x,a)}{v^-(t,x,\nu)-v^{\pm}(t,x,a)}\right)\mathrm{d}\nu \\
&-3\int_0^1\left(\frac{v^+_{xx}(t,x,\nu)-v^{\pm}_{xx}(t,x,a)}{v^+(t,x,\nu)-v^{\pm}(t,x,a)}\right)\left(\frac{v^+_x(t,x,\nu)-v^{\pm}_x(t,x,a)}{v^+(t,x,\nu)-v^{\pm}(t,x,a)}\right)\mathrm{d}\nu\Big) \\
&-\frac{1}{\varrho(v^{\pm}(t,x,a))}\int_0^1 \frac{\varrho(v^-(t,x,\nu))|\chi^+(v^-(t,x,\nu))|^{-2}\mathcal{R}^-_{xxx}(t,x,\nu)}{v^-(t,x,\nu)-v^{\pm}(t,x,a)}\mathrm{d}\nu \\
&+\frac{1}{\varrho(v^{\pm}(t,x,a))}\int_0^1 \frac{\varrho(v^+(t,x,\nu))|\chi^+(v^+(t,x,\nu))|^{-2}\mathcal{R}^+_{xxx}(t,x,\nu)}{v^+(t,x,\nu)-v^{\pm}(t,x,a)}\mathrm{d}\nu \\
&+\frac{1}{\varrho(v^{\pm}(t,x,a))}\int_0^1\int_0^1 \left\{\frac{\varrho(v^-(t,x,\nu))|\chi^+(v^-(t,x,\nu_1))|^{-2}}{(v^-(t,x,\nu_1)-v^{\pm}(t,x,a))}\right. \\
&\left.-\frac{\varrho(v^+(t,x,\nu))|\chi^+(v^+(t,x,\nu_1))|^{-2}}{(v^+(t,x,\nu_1)-v^{\pm}(t,x,a))}\right\} \\
&\times\left\{2\left[\left(\frac{v^-_x(t,x,\nu_2)-v^{\pm}_x(t,x,\nu_1)}{v^-(t,x,\nu_2)-v^{\pm}(t,x,\nu_1)}\right)^3-\left(\frac{v^+_x(t,x,\nu_2)-v^{\pm}_x(t,x,\nu_1)}{v^+(t,x,\nu_2)-v^{\pm}(t,x,\nu_1)}\right)^3\right]\right. \\
&-3\left[\left(\frac{v^-_{xx}(t,x,\nu_2)-v^{\pm}_{xx}(t,x,\nu_1)}{v^-(t,x,\nu_2)-v^{\pm}(t,x,\nu_1)}\right)\left(\frac{v^-_x(t,x,\nu_2)-v^{\pm}_x(t,x,\nu_1)}{v^-(t,x,\nu_2)-v^{\pm}(t,x,\nu_1)}\right)\right. \\
&\left.\left.-\left(\frac{v^+_{xx}(t,x,\nu_2)-v^{\pm}_{xx}(t,x,\nu_1)}{v^+(t,x,\nu_2)-v^{\pm}(t,x,\nu_1)}\right)\left(\frac{v^+_x(t,x,\nu_2)-v^{\pm}_x(t,x,\nu_1)}{v^+(t,x,\nu_2)-v^{\pm}(t,x,\nu_1)}\right)\right]\right\}\mathrm{d}\nu_2\mathrm{d}\nu_1,
\end{aligned}
\tag{81}
$$

where $|\chi^+(v^\pm(t,x,a))|^2 = |A(v^\pm(t,x,a))|^2 + \pi^2|\gamma^\pm(t,x,a)|^2$. In equations (76)–(81) the integrals must be evaluated in the sense of the principal value.

In order to estimate the derivatives of $v^\pm$ with respect to x, we will use the four following lemmas from singular integral equations and harmonic analysis theory.

Lemma 2. *Let Γ and $z(s)$ be, respectively, a rectifiable Jordan curve and its parametric representation. For all chord-arc (or Lavrentiev) curves $\Gamma : z = z(s)$, the corresponding Cauchy integral operator $\mathcal{S}_\Gamma$, defined by*

$$(\mathcal{S}_\Gamma\varphi)(s) = \text{p.v.} \int \frac{\varphi(\sigma)\mathrm{d}\sigma}{z(s) - z(\sigma)},$$

is bounded from $L^p(\mathrm{d}s)$ to $L^p(\mathrm{d}s)$, with $1 < p < \infty$ and we set $C_{\mathcal{S}_\Gamma} = \|\mathcal{S}_\Gamma\|_{\mathcal{L}(L^p,L^p)}$.

Proof. For the proof let us see [11,17,21,24,25,33,39,45–47,49,55]. In fact, the result in [24] is stronger than stated above. In [24] the author proved that the Cauchy integral operator $\mathcal{S}_\Gamma$ is bounded from L^p to L^p if and only if Γ is an Ahlfors or Carleson regular curve. A rectifiable Jordan curve Γ is said to be an Ahlfors or Carleson regular curve if there is a constant $C > 0$ such that for every $r > 0$ and every disc D with radius r, the length of $\Gamma \cap D$ is less than Cr. An example of an Ahlfors or Carleson regular curve is a Lavrentiev or chord-arc curve which is a rectifiable Jordan curve $z(s)$ for which there is a constant $C \geqq 1$ such that for all s_1, s_2, $|s_1 - s_2| \leqq C|z(s_1) - z(s_2)|$. A chord-arc curve is an Ahlfors or Carleson regular curve but not vice versa. In fact, for t and x fixed, the waterbag continuum $v^\pm(t,x)$ defines a chord-arc or Lavrentiev curve because $\gamma^\pm(t) \in L^\infty_{xa}$ and $W^{1,\infty}(\mathbb{R}^n) \hookrightarrow \mathscr{C}^{0,1}(\mathbb{R}^n)$. In fact, there exist two constants $C^\pm \geqq 1$ such that for all $a_1, a_2 \in [0,1]$, $|a_1 - a_2| \leqq C^\pm|v^\pm(a_1) - v^\pm(a_2)|$, where $C^\pm \leqq \|\gamma^\pm\|_{L^\infty_{txa}}$. Besides, the waterbag continuum $v^\pm(t,x)$ defines a Liapunov (or Lipschitz) curve (that is $\exists K^\pm > 0,\quad \forall a_1, a_2 \in [0,1],\quad |v^\pm(a_1) - v^\pm(a_2)| \leqq K^\pm|a_1 - a_2|$) because $\omega^\pm(t) \in L^\infty_{xa}$ and $W^{1,\infty}(\mathbb{R}^n) \hookrightarrow \mathscr{C}^{0,1}(\mathbb{R}^n)$. Moreover we have $K^\pm \leqq \|\omega^\pm\|_{L^\infty_{txa}}$. □

Lemma 3. *For all t fixed, let us suppose $f(t) = f(t,x,a) \in L^\infty_{xa}$, $\partial_a f(t) \in L^\infty_{xa}$ and positive integer k. Then the Calderón commutator*

$$(\mathcal{C}^{(k)}\varphi)(t,x,a) = \text{p.v.} \int_0^1 \frac{(f(t,x,a) - f(t,x,\nu))^k}{(a-\nu)^{k+1}} \varphi(x,\nu)\mathrm{d}\nu,$$

is bounded from L^p_{xa} to L^p_{xa}, with $1 < p < \infty$, for all t fixed. Moreover, we have

$$\|\mathcal{C}^{(k)}\varphi(t)\|_{L^p_{xa}} \leqq C^k \|\partial_a f(t)\|^k_{L^\infty_{xa}} \|\varphi(t)\|_{L^p_{xa}}.$$

Proof. For the proof, we refer to section 6 of chapter 9 in [46], or section 4 of chapter 7 in [55] or [11,22,26,50,51]. □

Lemma 4. *For all t fixed, let us suppose that $v^\pm(t)$, $\omega^\pm(t)$ and $\gamma^\pm(t)$ belong to L^∞_{xa}. Then the corresponding Cauchy integral operator $\mathcal{C}_{\Gamma^\pm}$, where*

$$(\mathcal{C}_{\Gamma^\pm}\varphi)(t,x,a) = \text{p.v.}\int_{\Gamma^\pm} \frac{\varphi(x,\nu)dv^\pm(t,x,\nu)}{v^\pm(t,x,\nu) - v^\pm(t,x,a)}$$
$$= \mp\text{p.v.}\int_0^1 \frac{\varphi(x,\nu)\omega^\pm(t,x,\nu)\mathrm{d}\nu}{v^\pm(t,x,\nu) - v^\pm(t,x,a)},$$

is bounded from L^p_{xa} to L^p_{xa}, with $1 < p < \infty$, for all t fixed. Moreover, we have

$$\|\mathcal{C}_{\Gamma^\pm}\|_{\mathcal{L}(L^p_{xa},L^p_{xa})} \leqq C_{\mathcal{S}_{\Gamma^\pm}} \|\omega^\pm(t)\|_{L^\infty_{xa}}.$$

Proof. Since $\gamma^\pm(t),\ \omega^\pm(t) \in L^\infty_{xa}$, the waterbag continuum constitutes a family of Liapunov (or Lipschitz) and Lavrentiev (or chord-arc) curves. Using Lemma 2, proof is straightforward. □

Lemma 5. *For all t fixed, let us suppose that $v^\pm(t), \omega^\pm(t)$ and $\gamma^\pm(t)$ belong to L^∞_{xa}. Then the operator*

$$(\mathcal{S}_{\Gamma^\pm\Gamma^\mp}\varphi)(t,x,a) = \int_{\Gamma^\pm} \frac{\varphi(x,\nu)\mathrm{d}v^\pm(t,x,\nu)}{v^\pm(t,x,\nu) - v^\mp(t,x,a)}$$
$$= \mp\int_0^1 \frac{\varphi(x,\nu)\omega^\pm(t,x,\nu)\mathrm{d}\nu}{v^\pm(t,x,\nu) - v^\mp(t,x,a)},$$

is bounded from L^p_{xa} to L^p_{xa}, with $1 < p < \infty$, for all t fixed. Moreover, we have

$$\|\mathcal{S}_{\Gamma^\pm\Gamma^\mp}\|_{\mathcal{L}(L^p_{xa},L^p_{xa})} \leqq C_{\mathcal{S}_\Gamma} \|\omega(t)\|_{L^\infty_{xa}},$$

where $\|\omega(t)\|_{L^\infty_{xa}} = \sup\left\{\|\omega^-(t)\|_{L^\infty_{xa}},\ \|\omega^+(t)\|_{L^\infty_{xa}}\right\}$.

Proof. Since for all $t, \omega^\pm(t) \in L^\infty_{xa}$, we can define $\widehat{\varphi}$ by $\widehat{\varphi}(x, v^\pm(t,x,\nu)) = \varphi(x,\nu)$. We next consider the function $\widetilde{\varphi}(x,\cdot)$ as the extension of $\widehat{\varphi}(x,\cdot) \in L^p(\Gamma^-)$ (resp. $\widehat{\varphi}(x,\cdot) \in L^p(\Gamma^+)$) on the curve Γ by setting it equal to zero on Γ^+ (resp. Γ^-). Since now the waterbag continuum constitutes a family of Liapunov (or Lipschitz) and Lavrentiev (or chord-arc) curves, using Lemma 2 we get

$$\|\mathcal{S}_{\Gamma^\pm\Gamma^\mp}\varphi\|^p_{L^p_{xa}} = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}a \left|\int_0^1 \frac{\varphi(x,\nu)\mathrm{d}v^\pm(t,x,\nu)}{v^\pm(t,x,\nu) - v^\mp(t,x,a)}\right|^p$$
$$= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}a \left|\int_0^1 \frac{\widehat{\varphi}(x,v^\pm(t,x,\nu))\mathrm{d}v^\pm(t,x,\nu)}{v^\pm(t,x,\nu) - v^\mp(t,x,a)}\right|^p$$
$$\leqq \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}a \left|\int_\Gamma \frac{\widetilde{\varphi}(x,\tau)\mathrm{d}\tau}{\tau - v}\right|^p$$
$$= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}a \left|\int_0^1 \frac{\widetilde{\varphi}(x,v(t,x,\nu))\mathrm{d}v(t,x,\nu)}{v(t,x,\nu) - v(t,x,a)}\right|^p$$
$$\leqq C^p_{\mathcal{S}_\Gamma} \|\omega(t)\|^p_{L^\infty_{xa}} \|\varphi(t)\|^p_{L^p_{xa}}.$$

□

Before going further we note the two following inequalities which we will use many times in the sequel. We first have

$$\frac{|A(v^{\pm}(t,x,a))|}{|\chi^{+}(v^{\pm}(t,x,a))|^2} \leqq \frac{|\chi^{+}(v^{\pm}(t,x,a))|}{|\chi^{+}(v^{\pm}(t,x,a))|^2} \leqq |\omega^{\pm}(t,x,a)|, \tag{82}$$

and

$$\frac{1}{|\chi^{+}(v^{\pm}(t,x,a))|^2} \leqq \frac{1}{\pi^2|\gamma^{\pm}(t,x,a)|^2} \leqq |\omega^{\pm}(t,x,a)|^2. \tag{83}$$

Using the properties (32)–(34) resulting from the study of the eigenvalue problem in Section 3.2, there exist two constants $\kappa > 0$, and $\mathcal{K} > 0$ such that

$$\kappa^{-1} \leqq |\varrho(v^{\pm}(t,x,a))| \leqq \mathcal{K}, \tag{84}$$

so that we get

$$\begin{aligned} &\|P_1(v^{\pm})\|_{L^p_{xa}} \leqq C(\alpha_0,\alpha_1)\|v^{\pm}\|_{L^\infty_{xa}}, \quad \text{and} \\ &\left\|\frac{P_1(v^{\pm})}{\varrho(v^{\pm})}\right\|_{L^p_{xa}} \leqq C(\alpha_0,\alpha_1,\kappa)\|v^{\pm}\|_{L^\infty_{xa}}. \end{aligned} \tag{85}$$

L^2-estimate for $v^{\pm}_x$

Using inequality (82) the first term of the right-hand side of (80) is bounded as

$$\left\|\frac{|A(v^{\pm}(t,x,a))|\mathcal{R}^{\pm}_x(t,x,a)}{|\chi^{+}(v^{\pm}(t,x,a))|^2}\right\|_{L^2_{xa}} \leqq \|\omega^{\pm}\|_{L^\infty_{xa}}\|\mathcal{R}^{\pm}_x\|_{L^2_{xa}}. \tag{86}$$

Using inequalities (83)–(84), Lemmas 4 and 5 we get

$$\begin{aligned} &\left\|\frac{1}{\varrho(v^{\pm}(t,x,a))}\int_0^1 \frac{\varrho(v^{+}(t,x,\nu))|\chi^{+}(v^{+}(t,x,\nu))|^{-2}\mathcal{R}^{+}_x(t,x,\nu)}{v^{+}(t,x,\nu)-v^{\pm}(t,x,a)}\mathrm{d}\nu\right\|_{L^2_{xa}} \\ &\quad \leqq C(\kappa,\mathcal{K})\|\omega^{+}\|_{L^\infty_{xa}}\|\omega\|_{L^\infty_{xa}}\|\mathcal{R}^{+}_x\|_{L^2_{xa}}, \end{aligned} \tag{87}$$

and

$$\begin{aligned} &\left\|\frac{1}{\varrho(v^{\pm}(t,x,a))}\int_0^1 \frac{\varrho(v^{-}(t,x,\nu))|\chi^{+}(v^{-}(t,x,\nu))|^{-2}\mathcal{R}^{-}_x(t,x,\nu)}{v^{-}(t,x,\nu)-v^{\pm}(t,x,a)}\mathrm{d}\nu\right\|_{L^2_{xa}} \\ &\quad \leqq C(\kappa,\mathcal{K})\|\omega^{-}\|_{L^\infty_{xa}}\|\omega\|_{L^\infty_{xa}}\|\mathcal{R}^{-}_x\|_{L^2_{xa}}. \end{aligned} \tag{88}$$

L^p-estimate for $v^{\pm}_{xx}$

As we have done for the estimate of $v^{\pm}_x$, we get for $1 < p < \infty$,

$$\left\|\frac{|A(v^{\pm}(t,x,a))|\mathcal{R}^{\pm}_{xx}(t,x,a)}{|\chi^{+}(v^{\pm}(t,x,a))|^2}\right\|_{L^p_{xa}} \leqq \|\omega^{\pm}\|_{L^\infty_{xa}}\|\mathcal{R}^{\pm}_{xx}\|_{L^p_{xa}}, \tag{89}$$

$$\begin{aligned} &\left\|\frac{1}{\varrho(v^{\pm}(t,x,a))}\int_0^1 \frac{\varrho(v^{+}(t,x,\nu))|\chi^{+}(v^{+}(t,x,\nu))|^{-2}\mathcal{R}^{+}_{xx}(t,x,\nu)}{(v^{+}(t,x,\nu)-v^{\pm}(t,x,a)}\mathrm{d}\nu\right\|_{L^p_{xa}} \\ &\quad \leqq C(\kappa,\mathcal{K})\|\omega^{+}\|_{L^\infty_{xa}}\|\omega\|_{L^\infty_{xa}}\|\mathcal{R}^{+}_{xx}\|_{L^p_{xa}}, \end{aligned} \tag{90}$$

and

$$\left\| \frac{1}{\varrho(v^{\pm}(t,x,a))} \int_0^1 \frac{\varrho(v^-(t,x,\nu))|\chi^+(v^-(t,x,\nu))|^{-2}\mathcal{R}_{xx}^-(t,x,\nu)}{v^-(t,x,\nu) - v^{\pm}(t,x,a)} \mathrm{d}\nu \right\|_{L_{xa}^p}$$
$$\leqq C(\kappa,\mathcal{K})\|\omega^-\|_{L_{xa}^\infty}\|\omega\|_{L_{xa}^\infty}\|\mathcal{R}_{xx}^-\|_{L_{xa}^p}. \tag{91}$$

Let us set now

$$\mathcal{T}_1^{-,-} = \frac{A(v^{\pm}(t,x,a))}{|\chi^+(v^{\pm}(t,x,a))|^2} \int_0^1 \left(\frac{v_x^-(t,x,\nu) - v_x^-(t,x,a)}{v^-(t,x,\nu) - v^-(t,x,a)} \right)^2 \mathrm{d}\nu, \tag{92}$$

and

$$\mathcal{T}_1^{+,+} = \frac{A(v^{\pm}(t,x,a))}{|\chi^+(v^{\pm}(t,x,a))|^2} \int_0^1 \left(\frac{v_x^+(t,x,\nu) - v_x^+(t,x,a)}{v^+(t,x,\nu) - v^+(t,x,a)} \right)^2 \mathrm{d}\nu. \tag{93}$$

As we have $W^{1,\infty}(\mathbb{R}^n) \hookrightarrow \mathscr{C}^{0,1}(\mathbb{R}^n)$ we get

$$|\mathcal{T}_1^{-,-}| \leqq \frac{|A(v^{\pm}(t,x,a))|}{|\chi^+(v^{\pm}(t,x,a))|^2} \int_0^1 \left(\frac{v_x^-(t,x,\nu) - v_x^-(t,x,a)}{\nu - a} \right)^2$$
$$\times \left(\frac{v^-(t,x,\nu) - v^-(t,x,a)}{\nu - a} \right)^{-2} \mathrm{d}\nu \leqq \|\omega^{\pm}\|_{L_{xa}^\infty}\|\omega_x^-\|_{L_{xa}^\infty}^2\|\gamma^-\|_{L_{xa}^\infty}^2.$$

Therefore we obtain

$$\|\mathcal{T}_1^{-,-}\|_{L_{xa}^p} \leqq \|\omega^{\pm}\|_{L_{xa}^\infty}\|\omega_x^-\|_{L_{xa}^\infty}^2\|\gamma^-\|_{L_{xa}^\infty}^2. \tag{94}$$

In the same way we get

$$\|\mathcal{T}_1^{+,+}\|_{L_{xa}^p} \leqq \|\omega^{\pm}\|_{L_{xa}^\infty}\|\omega_x^+\|_{L_{xa}^\infty}^2\|\gamma^+\|_{L_{xa}^\infty}^2. \tag{95}$$

Let us now set

$$\mathcal{T}_1^{-,+} = \frac{A(v^{\pm}(t,x,a))}{|\chi^+(v^{\pm}(t,x,a))|^2} \int_0^1 \left(\frac{v_x^-(t,x,\nu) - v_x^+(t,x,a)}{v^-(t,x,\nu) - v^+(t,x,a)} \right)^2 \mathrm{d}\nu, \tag{96}$$

and

$$\mathcal{T}_1^{+,-} = \frac{A(v^{\pm}(t,x,a))}{|\chi^+(v^{\pm}(t,x,a))|^2} \int_0^1 \left(\frac{v_x^+(t,x,\nu) - v_x^-(t,x,a)}{v^+(t,x,\nu) - v^-(t,x,a)} \right)^2 \mathrm{d}\nu. \tag{97}$$

By noting that $v^-(t,x,\nu) - v^-(t,x,1) \leqq 0$ and $v^+(t,x,1) - v^+(t,x,a) \leqq 0$ and by assuming that $v_x^+(t,x,1) = v_x^-(t,x,1)$ (recall that $v^+(t,x,1) = v^-(t,x,1)$), then using the embedding $W^{1,\infty}(\mathbb{R}^n) \hookrightarrow \mathscr{C}^{0,1}(\mathbb{R}^n)$ we have the following estimate

$$\begin{aligned}
&\left|\frac{v_x^-(t,x,\nu)-v_x^+(t,x,a)}{v^-(t,x,\nu)-v^+(t,x,a)}\right| \\
&\quad = \left|\frac{v_x^-(t,x,\nu)-v_x^-(t,x,1)+v_x^+(t,x,1)-v_x^+(t,x,a)}{v^-(t,x,\nu)-v^-(t,x,1)+v^+(t,x,1)-v^+(t,x,a)}\right| \\
&\quad \leqq \left|\frac{v_x^-(t,x,\nu)-v_x^-(t,x,1)}{v^-(t,x,\nu)-v^-(t,x,1)+v^+(t,x,1)-v^+(t,x,a)}\right| \\
&\qquad + \left|\frac{v_x^+(t,x,1)-v_x^+(t,x,a)}{v^-(t,x,\nu)-v^-(t,x,1)+v^+(t,x,1)-v^+(t,x,a)}\right| \\
&\quad \leqq \left|\frac{v_x^-(t,x,\nu)-v_x^-(t,x,1)}{v^-(t,x,\nu)-v^-(t,x,1)}\right| + \left|\frac{v_x^+(t,x,1)-v_x^+(t,x,a)}{v^+(t,x,1)-v^+(t,x,a)}\right| \\
&\quad \leqq \|\omega_x^+\|_{L^\infty_{xa}}\|\gamma^+\|_{L^\infty_{xa}} + \|\omega_x^-\|_{L^\infty_{xa}}\|\gamma^-\|_{L^\infty_{xa}}.
\end{aligned} \tag{98}$$

Using (98) we obtain

$$\|\mathcal{T}_1^{-,+}\|_{L^p_{xa}} \leqq \|\omega^\pm\|_{L^\infty_{xa}}\left(\|\omega_x^-\|_{L^\infty_{xa}}\|\gamma^-\|_{L^\infty_{xa}} + \|\omega_x^+\|_{L^\infty_{xa}}\|\gamma^+\|_{L^\infty_{xa}}\right)^2, \tag{99}$$

and

$$\|\mathcal{T}_1^{+,-}\|_{L^p_{xa}} \leqq \|\omega^\pm\|_{L^\infty_{xa}}\left(\|\omega_x^-\|_{L^\infty_{xa}}\|\gamma^-\|_{L^\infty_{xa}} + \|\omega_x^+\|_{L^\infty_{xa}}\|\gamma^+\|_{L^\infty_{xa}}\right)^2. \tag{100}$$

Let us now define

$$\begin{aligned}
\mathbb{T}_1^\pm = \frac{1}{\varrho(v^\pm(t,x,a))}\int_0^1 \Bigg[&\frac{\varrho(v^-(t,x,\nu))|\chi^+(v^-(t,x,\nu_1))|^{-2}}{v^-(t,x,\nu_1)-v^\pm(t,x,a)} \\
&-\frac{\varrho(v^+(t,x,\nu))|\chi^+(v^+(t,x,\nu_1))|^{-2}}{v^+(t,x,\nu_1)-v^\pm(t,x,a)}\Bigg]\left[f_-^\pm(t,x,\nu_1)-f_+^\pm(t,x,\nu_1)\right]\mathrm{d}\nu_1,
\end{aligned}$$

where

$$f_-^\pm(t,x,\nu_1) = \int_0^1\left(\frac{v_x^-(t,x,\nu_2)-v_x^\pm(t,x,\nu_1)}{v^-(t,x,\nu_2)-v^\pm(t,x,\nu_1)}\right)^2\mathrm{d}\nu_2,$$

$$f_+^\pm(t,x,\nu_1) = \int_0^1\left(\frac{v_x^+(t,x,\nu_2)-v_x^\pm(t,x,\nu_1)}{v^+(t,x,\nu_2)-v^\pm(t,x,\nu_1)}\right)^2\mathrm{d}\nu_2.$$

Using inequalities (83)–(84), Lemmas 4 and 5 we get

$$\|\mathbb{T}_1^\pm\|_{L^p_{xa}} \leqq C(\kappa,\mathcal{K})\|\omega\|_{L^\infty_{xa}}\left(\|\omega^-\|_{L^\infty_{xa}}+\|\omega^+\|_{L^\infty_{xa}}\right)\left(\|f_-^\pm\|_{L^p_{xa}}+\|f_+^\pm\|_{L^p_{xa}}\right). \tag{101}$$

As we have done to estimate the term $\mathcal{T}_1^{\pm,\pm}$, we obtain

$$\|f_-^+\|_{L^p_{xa}} \leqq \left(\|\omega_x^-\|_{L^\infty_{xa}}\|\gamma^-\|_{L^\infty_{xa}} + \|\omega_x^+\|_{L^\infty_{xa}}\|\gamma^+\|_{L^\infty_{xa}}\right)^2 \tag{102}$$

$$\|f_+^-\|_{L^p_{xa}} \leqq \left(\|\omega_x^-\|_{L^\infty_{xa}}\|\gamma^-\|_{L^\infty_{xa}} + \|\omega_x^+\|_{L^\infty_{xa}}\|\gamma^+\|_{L^\infty_{xa}}\right)^2 \tag{103}$$

$$\|f_-^-\|_{L^p_{xa}} \leqq \|\omega_x^-\|^2_{L^\infty_{xa}}\|\gamma^-\|^2_{L^\infty_{xa}} \tag{104}$$

$$\|f_+^+\|_{L^p_{xa}} \leqq \|\omega_x^+\|^2_{L^\infty_{xa}}\|\gamma^+\|^2_{L^\infty_{xa}}. \tag{105}$$

Using (102)–(105), estimate (101) becomes

$$\|\mathbb{T}_1^{\pm}\|_{L^p_{xa}} \leqq C(\kappa, \mathcal{K})\|\omega\|_{L^\infty_{xa}} \left(\|\omega^-\|_{L^\infty_{xa}} + \|\omega^+\|_{L^\infty_{xa}}\right) \left(\|\omega_x^-\|_{L^\infty_{xa}} \|\gamma^-\|_{L^\infty_{xa}} + \|\omega_x^+\|_{L^\infty_{xa}} \|\gamma^+\|_{L^\infty_{xa}}\right)^2, \tag{106}$$

which ends the L^p-estimate of $v_{xx}^{\pm}$.

<u>L^2-estimate for $v_{xxx}^{\pm}$</u>

In the same way we have estimated the term $v_{xx}^{\pm}$, we get

$$\left\| \frac{|A(v^{\pm}(t,x,a))|\mathcal{R}_{xxx}^{\pm}(t,x,a)}{|\chi^+(v^{\pm}(t,x,a))|^2} \right\|_{L^2_{xa}} \leqq \|\omega^{\pm}\|_{L^\infty_{xa}} \|\mathcal{R}_{xxx}^{\pm}\|_{L^2_{xa}}, \tag{107}$$

$$\left\| \frac{1}{\varrho(v^{\pm}(t,x,a))} \int_0^1 \frac{\varrho(v^+(t,x,\nu))|\chi^+(v^+(t,x,\nu))|^{-2}\mathcal{R}_{xxx}^+(t,x,\nu)}{v^+(t,x,\nu) - v^{\pm}(t,x,a)} \mathrm{d}\nu \right\|_{L^2_{xa}} \leqq C(\kappa, \mathcal{K})\|\omega^+\|_{L^\infty_{xa}} \|\omega\|_{L^\infty_{xa}} \|\mathcal{R}_{xxx}^+\|_{L^2_{xa}}, \tag{108}$$

$$\left\| \frac{1}{\varrho(v^{\pm}(t,x,a))} \int_0^1 \frac{\varrho(v^-(t,x,\nu))|\chi^+(v^-(t,x,\nu))|^{-2}\mathcal{R}_{xxx}^-(t,x,\nu)}{v^-(t,x,\nu) - v^{\pm}(t,x,a)} \mathrm{d}\nu \right\|_{L^2_{xa}} \leqq C(\kappa, \mathcal{K})\|\omega^-\|_{L^\infty_{xa}} \|\omega\|_{L^\infty_{xa}} \|\mathcal{R}_{xxx}^-\|_{L^2_{xa}}, \tag{109}$$

and

$$\left\| \frac{A(v^{\pm}(t,x,a))}{|\chi^+(v^{\pm}(t,x,a))|^2} \left\{ \int_0^1 \left(\frac{v_x^-(t,x,\nu) - v_x^{\pm}(t,x,a)}{v^-(t,x,\nu) - v^{\pm}(t,x,a)} \right)^3 \mathrm{d}\nu - \int_0^1 \left(\frac{v_x^+(t,x,\nu) - v_x^{\pm}(t,x,a)}{v^+(t,x,\nu) - v^{\pm}(t,x,a)} \right)^3 \mathrm{d}\nu \right\} \right\|_{L^2_{xa}} \leqq \|\omega^{\pm}\|_{L^\infty_{xa}} \left(\|\omega_x^+\|^3_{L^\infty_{xa}} \|\gamma^+\|^3_{L^\infty_{xa}} + \|\omega_x^-\|^3_{L^\infty_{xa}} \|\gamma^-\|^3_{L^\infty_{xa}} \right). \tag{110}$$

Let us now set

$$\mathcal{T}_2^{-,-} = \frac{A(v^{\pm}(t,x,a))}{|\chi^+(v^{\pm}(t,x,a))|^2} \int_0^1 \left(\frac{v_{xx}^-(t,x,\nu) - v_{xx}^-(t,x,a)}{v^-(t,x,\nu) - v^-(t,x,a)} \right) \times \left(\frac{v_x^-(t,x,\nu) - v_x^-(t,x,a)}{v^-(t,x,\nu) - v^-(t,x,a)} \right) \mathrm{d}\nu, \tag{111}$$

and

$$\mathcal{T}_2^{+,+} = \frac{A(v^{\pm}(t,x,a))}{|\chi^+(v^{\pm}(t,x,a))|^2} \int_0^1 \left(\frac{v_{xx}^+(t,x,\nu) - v_{xx}^+(t,x,a)}{v^+(t,x,\nu) - v^+(t,x,a)} \right) \times \left(\frac{v_x^+(t,x,\nu) - v_x^+(t,x,a)}{v^+(t,x,\nu) - v^+(t,x,a)} \right) \mathrm{d}\nu. \tag{112}$$

Let us start with $\mathcal{T}_2^{-,-}$. The term $\mathcal{T}_2^{-,-}$ can be decomposed as

$$\mathcal{T}_2^{-,-} = \mathcal{T}_{21}^{-,-} - \mathcal{T}_{22}^{-,-}, \tag{113}$$

where

$$\mathcal{T}_{21}^{-,-} = \frac{A(v^{\pm}(t,x,a))}{|\chi^{+}(v^{\pm}(t,x,a))|^2}\int_0^1 \frac{v_{xx}^{-}(t,x,\nu)\left(v_x^{-}(t,x,\nu)-v_x^{-}(t,x,a)\right)}{\left(v^{-}(t,x,\nu)-v^{-}(t,x,a)\right)^2}\mathrm{d}\nu$$

$$= \frac{A(v^{\pm}(t,x,a))}{|\chi^{+}(v^{\pm}(t,x,a))|^2}\int_0^1 \frac{v_{xx}^{-}(t,x,\nu)\left(v_x^{-}(t,x,\nu)-v_x^{-}(t,x,a)\right)}{(\nu-a)^2} \times\left(\frac{v^{-}(t,x,\nu)-v^{-}(t,x,a)}{\nu-a}\right)^{-2}\mathrm{d}\nu,$$

and

$$\mathcal{T}_{22}^{-,-} = \frac{A(v^{\pm}(t,x,a))}{|\chi^{+}(v^{\pm}(t,x,a))|^2}v_{xx}^{-}(t,x,a)\int_0^1 \left(\frac{v_x^{-}(t,x,\nu)-v_x^{-}(t,x,a)}{v^{-}(t,x,\nu)-v^{-}(t,x,a)}\right) \times\frac{(\gamma^{-}\omega^{-})(t,x,\nu)\mathrm{d}\nu}{v^{-}(t,x,\nu)-v^{-}(t,x,a)}.$$

Using Lemma 3 we get

$$\|\mathcal{T}_{21}^{-,-}\|_{L^2_{xa}} \leqq C\|\omega^{\pm}\|_{L^\infty_{xa}}\|\omega_x^{-}\|_{L^\infty_{xa}}\|\gamma^{-}\|^2_{L^\infty_{xa}}\|v_{xx}^{-}\|_{L^2_{xa}}.$$

Using Lemma 4 and Young's inequality we get

$$\|\mathcal{T}_{22}^{-,-}\|_{L^2_{xa}} \leqq C\|\omega^{\pm}\|_{L^\infty_{xa}}\|\omega^{-}\|_{L^\infty_{xa}}\|v_{xx}^{-}\|_{L^4_{xa}}\|\omega_x^{-}\|_{L^\infty_{xa}}\|\gamma^{-}\|_{L^\infty_{xa}}\|\gamma^{-}\|_{L^4_{xa}}.$$

Therefore, we obtain for the term $\mathcal{T}_2^{-,-}$ the following bound

$$\|\mathcal{T}_{2}^{-,-}\|_{L^2_{xa}} \leqq C\|\omega^{\pm}\|_{L^\infty_{xa}}\|\omega_x^{-}\|_{L^\infty_{xa}}\|\gamma^{-}\|^2_{L^\infty_{xa}}\|\omega^{-}\|_{L^\infty_{xa}}\left(\|v_{xx}^{-}\|_{L^2_{xa}}+\|v_{xx}^{-}\|_{L^4_{xa}}\right). \tag{114}$$

In same way, we obtain for the term $\mathcal{T}_2^{+,+}$ the bound

$$\|\mathcal{T}_{2}^{+,+}\|_{L^2_{xa}} \leqq C\|\omega^{\pm}\|_{L^\infty_{xa}}\|\omega_x^{+}\|_{L^\infty_{xa}}\|\gamma^{+}\|^2_{L^\infty_{xa}}\|\omega^{+}\|_{L^\infty_{xa}}\left(\|v_{xx}^{+}\|_{L^2_{xa}}+\|v_{xx}^{+}\|_{L^4_{xa}}\right). \tag{115}$$

The term $\|v_{xx}^{\pm}\|_{L^4_{xa}}$ in (114) and (115) is bounded using L^p-estimate for $v_{xx}^{\pm}$ proved above, in which we replace $\|\mathcal{R}_{xx}^{\pm}\|_{L^4_{xa}}$ by $\|\mathcal{R}_{xx}^{\pm}\|_{H^1_{xa}}$ because of the Sobolev embedding $L^4(\mathbb{R}^2) \hookrightarrow H^{1/2}(\mathbb{R}^2)$. Let us now set

$$\mathcal{T}_{2}^{-,+} = \frac{A(v^{\pm}(t,x,a))}{|\chi^{+}(v^{\pm}(t,x,a))|^2}\int_0^1 \left(\frac{v_{xx}^{-}(t,x,\nu)-v_{xx}^{+}(t,x,a)}{v^{-}(t,x,\nu)-v^{+}(t,x,a)}\right) \times\left(\frac{v_x^{-}(t,x,\nu)-v_x^{+}(t,x,a)}{v^{-}(t,x,\nu)-v^{+}(t,x,a)}\right)\mathrm{d}\nu, \tag{116}$$

and

$$\mathcal{T}_{2}^{+,-} = \frac{A(v^{\pm}(t,x,a))}{|\chi^{+}(v^{\pm}(t,x,a))|^2}\int_0^1 \left(\frac{v_{xx}^{+}(t,x,\nu)-v_{xx}^{-}(t,x,a)}{v^{+}(t,x,\nu)-v^{-}(t,x,a)}\right) \times\left(\frac{v_x^{+}(t,x,\nu)-v_x^{-}(t,x,a)}{v^{+}(t,x,\nu)-v^{-}(t,x,a)}\right)\mathrm{d}\nu. \tag{117}$$

Let us start with $\mathcal{T}_2^{-,+}$. The term $\mathcal{T}_2^{-,+}$ can be decomposed as

$$\mathcal{T}_2^{-,+} = \mathcal{T}_{21}^{-,+} - \mathcal{T}_{22}^{-,+}, \tag{118}$$

where

$$\mathcal{T}_{21}^{-,+} = \frac{A(v^{\pm}(t,x,a))}{|\chi^{+}(v^{\pm}(t,x,a))|^2} \int_0^1 \frac{(\gamma^{-}\omega^{-}v_{xx}^{-})(t,x,\nu)}{v^{-}(t,x,\nu) - v^{+}(t,x,a)} \times \left(\frac{v_x^{-}(t,x,\nu) - v_x^{+}(t,x,a)}{v^{-}(t,x,\nu) - v^{+}(t,x,a)} \right) \mathrm{d}\nu,$$

and

$$\mathcal{T}_{22}^{-,+} = \frac{A(v^{\pm}(t,x,a))}{|\chi^{+}(v^{\pm}(t,x,a))|^2} v_{xx}^{-}(t,x,a) \int_0^1 \frac{(\gamma^{-}\omega^{-})(t,x,\nu)}{v^{-}(t,x,\nu) - v^{+}(t,x,a)} \times \left(\frac{v_x^{-}(t,x,\nu) - v_x^{+}(t,x,a)}{v^{-}(t,x,\nu) - v^{+}(t,x,a)} \right) \mathrm{d}\nu.$$

Using Lemma 5, inequality (98) and Young's inequality we get

$$\|\mathcal{T}_{21}^{-,+}\|_{L^2_{xa}} \leqq C \|\omega^{\pm}\|_{L^\infty_{xa}} \|\gamma^{-}\|_{L^\infty_{xa}} \|v_{xx}^{-}\|_{L^2_{xa}} \|\omega\|_{L^\infty_{xa}} \Big(\|\omega_x^{+}\|_{L^\infty_{xa}} \|\gamma^{+}\|_{L^\infty_{xa}} + \|\omega_x^{-}\|_{L^\infty_{xa}} \|\gamma^{-}\|_{L^\infty_{xa}} \Big),$$

and

$$\|\mathcal{T}_{22}^{-,+}\|_{L^2_{xa}} \leqq C \|\omega^{\pm}\|_{L^\infty_{xa}} \|\gamma^{-}\|_{L^4_{xa}} \|v_{xx}^{-}\|_{L^4_{xa}} \|\omega\|_{L^\infty_{xa}} \times \Big(\|\omega_x^{+}\|_{L^\infty_{xa}} \|\gamma^{+}\|_{L^\infty_{xa}} + \|\omega_x^{-}\|_{L^\infty_{xa}} \|\gamma^{-}\|_{L^\infty_{xa}} \Big).$$

Therefore, we obtain for the term $\mathcal{T}_2^{-,+}$ the following bound

$$\|\mathcal{T}_2^{-,+}\|_{L^2_{xa}} \leqq C \|\omega^{\pm}\|_{L^\infty_{xa}} \|\omega_x^{-}\|_{L^\infty_{xa}} \|\omega\|_{L^\infty_{xa}} \Big(\|\omega_x^{+}\|_{L^\infty_{xa}} \|\gamma^{+}\|_{L^\infty_{xa}} + \|\omega_x^{-}\|_{L^\infty_{xa}} \|\gamma^{-}\|_{L^\infty_{xa}} \Big) \Big(\|v_{xx}^{-}\|_{L^2_{xa}} + \|v_{xx}^{-}\|_{L^4_{xa}} \Big). \tag{119}$$

In same way, we obtain for the term $\mathcal{T}_2^{+,-}$ the bound

$$\|\mathcal{T}_2^{+,-}\|_{L^2_{xa}} \leqq C \|\omega^{\pm}\|_{L^\infty_{xa}} \|\omega_x^{-}\|_{L^\infty_{xa}} \|\omega\|_{L^\infty_{xa}} \times \Big(\|\omega_x^{+}\|_{L^\infty_{xa}} \|\gamma^{+}\|_{L^\infty_{xa}} + \|\omega_x^{-}\|_{L^\infty_{xa}} \|\gamma^{-}\|_{L^\infty_{xa}} \Big) \Big(\|v_{xx}^{-}\|_{L^2_{xa}} + \|v_{xx}^{-}\|_{L^4_{xa}} \Big). \tag{120}$$

The term $\|v_{xx}^{\pm}\|_{L^4_{xa}}$ in (119) and (120) is bounded using L^p-estimate for $v_{xx}^{\pm}$ proved above, in which we replace $\|\mathcal{R}_{xx}^{\pm}\|_{L^4_{xa}}$ by $\|\mathcal{R}_{xx}^{\pm}\|_{H^1_{xa}}$ because of the Sobolev embedding $L^4(\mathbb{R}^2) \hookrightarrow H^{1/2}(\mathbb{R}^2)$. Let us now define

$$\mathbb{T}_2^{\pm} = \frac{1}{\varrho(v^{\pm}(t,x,a))} \int_0^1 \left[\frac{\varrho(v^{-}(t,x,\nu))|\chi^{+}(v^{-}(t,x,\nu_1))|^{-2}}{v^{-}(t,x,\nu_1) - v^{\pm}(t,x,a)} - \frac{\varrho(v^{+}(t,x,\nu))|\chi^{+}(v^{+}(t,x,\nu_1))|^{-2}}{v^{+}(t,x,\nu_1) - v^{\pm}(t,x,a)} \right] \times \left[f_{-}^{\pm}(t,x,\nu_1) - f_{+}^{\pm}(t,x,\nu_1) - g_{-}^{\pm}(t,x,\nu_1) + g_{+}^{\pm}(t,x,\nu_1) \right] \mathrm{d}\nu_1, \tag{121}$$

where

$$f_-^{\pm}(t,x,v_1) = 2\int_0^1 \left(\frac{v_x^-(t,x,v_2) - v_x^{\pm}(t,x,v_1)}{v^-(t,x,v_2) - v^{\pm}(t,x,v_1)}\right)^3 \mathrm{d}v_2,$$

$$f_+^{\pm}(t,x,v_1) = 2\int_0^1 \left(\frac{v_x^+(t,x,v_2) - v_x^{\pm}(t,x,v_1)}{v^+(t,x,v_2) - v^{\pm}(t,x,v_1)}\right)^3 \mathrm{d}v_2,$$

$$g_-^{\pm}(t,x,v_1) = 3\int_0^1 \left(\frac{v_{xx}^-(t,x,v_2) - v_{xx}^{\pm}(t,x,v_1)}{v^-(t,x,v_2) - v^{\pm}(t,x,v_1)}\right) \times \left(\frac{v_x^-(t,x,v_2) - v_x^{\pm}(t,x,v_1)}{v^-(t,x,v_2) - v^{\pm}(t,x,v_1)}\right) \mathrm{d}v_2,$$

$$g_+^{\pm}(t,x,v_1) = 3\int_0^1 \left(\frac{v_{xx}^+(t,x,v_2) - v_{xx}^{\pm}(t,x,v_1)}{v^+(t,x,v_2) - v^{\pm}(t,x,v_1)}\right) \times \left(\frac{v_x^+(t,x,v_2) - v_x^{\pm}(t,x,v_1)}{v^+(t,x,v_2) - v^{\pm}(t,x,v_1)}\right) \mathrm{d}v_2.$$

Using inequality (83)–(84), Lemmas 4 and 5 we get

$$\|\mathbb{T}_2^{\pm}\|_{L^2_{xa}} \leqq C(\kappa,\mathcal{K})\|\omega\|_{L^\infty_{xa}} \left(\|\omega^-\|_{L^\infty_{xa}} + \|\omega^+\|_{L^\infty_{xa}}\right)\left(\|f_-^{\pm}\|_{L^2_{xa}} + \|f_+^{\pm}\|_{L^2_{xa}} + \|g_-^{\pm}\|_{L^2_{xa}} + \|g_+^{\pm}\|_{L^2_{xa}}\right). \tag{122}$$

In the same way we have bounded the term $\mathcal{T}_2^{\pm,\pm}$, we obtain

$$\|f_-^-\|_{L^2_{xa}} \leqq \|\omega_x^-\|^3_{L^\infty_{xa}}\|\gamma^-\|^3_{L^\infty_{xa}},$$

$$\|f_+^+\|_{L^2_{xa}} \leqq \|\omega_x^+\|^3_{L^\infty_{xa}}\|\gamma^+\|^3_{L^\infty_{xa}},$$

$$\|f_-^+\|_{L^2_{xa}} \leqq \left(\|\omega_x^-\|_{L^\infty_{xa}}\|\gamma^-\|_{L^\infty_{xa}} + \|\omega_x^+\|_{L^\infty_{xa}}\|\gamma^+\|_{L^\infty_{xa}}\right)^3,$$

$$\|f_+^-\|_{L^2_{xa}} \leqq \left(\|\omega_x^-\|_{L^\infty_{xa}}\|\gamma^-\|_{L^\infty_{xa}} + \|\omega_x^+\|_{L^\infty_{xa}}\|\gamma^+\|_{L^\infty_{xa}}\right)^3,$$

$$\|g_-^-\|_{L^2_{xa}} \leqq C\|\omega^-\|_{L^\infty_{xa}}\|\omega_x^-\|_{L^\infty_{xa}}\|\gamma^-\|^2_{L^\infty_{xa}}\left(\|v_{xx}^-\|_{L^2_{xa}} + \|v_{xx}^-\|_{L^4_{xa}}\right),$$

$$\|g_+^+\|_{L^2_{xa}} \leqq C\|\omega^+\|_{L^\infty_{xa}}\|\omega_x^+\|_{L^\infty_{xa}}\|\gamma^+\|^2_{L^\infty_{xa}}\left(\|v_{xx}^+\|_{L^2_{xa}} + \|v_{xx}^+\|_{L^4_{xa}}\right),$$

$$\|g_-^+\|_{L^2_{xa}} \leqq C\|\omega\|_{L^\infty_{xa}}\|\omega_x^-\|_{L^\infty_{xa}}\left(\|\omega_x^+\|_{L^\infty_{xa}}\|\gamma^+\|_{L^\infty_{xa}} + \|\omega_x^-\|_{L^\infty_{xa}}\|\gamma^-\|_{L^\infty_{xa}}\right) \times \left(\|v_{xx}^-\|_{L^2_{xa}} + \|v_{xx}^-\|_{L^4_{xa}}\right),$$

$$\|g_+^-\|_{L^2_{xa}} \leqq C\|\omega\|_{L^\infty_{xa}}\|\omega_x^-\|_{L^\infty_{xa}}\left(\|\omega_x^+\|_{L^\infty_{xa}}\|\gamma^+\|_{L^\infty_{xa}} + \|\omega_x^-\|_{L^\infty_{xa}}\|\gamma^-\|_{L^\infty_{xa}}\right) \times \left(\|v_{xx}^+\|_{L^2_{xa}} + \|v_{xx}^+\|_{L^4_{xa}}\right),$$

which ends the L^2-estimate of $v_{xxx}^{\pm}$.

4.4.6. A priori estimates for $k^{\pm}$ From the study of the eigenvalue problem in Section 3.2, in which we have obtained some properties (32)–(34) for the characteristic equation (30) and its derivative (31), we know that there exist three constants $\mathcal{K}_1 > 0$, $\mathcal{K}_2 > 0$ and $\mathcal{K}_3 > 0$, independent of the variables (t, x), such that for all $x \in \mathbb{R}/\mathbb{Z}$ and for t fixed we have

$$|k^{\pm}| \leqq \mathcal{K}_1, \tag{123}$$

$$|\chi'(k^{\pm})| \geqq \mathcal{K}_2^{-1}, \quad \text{and} \quad |v_b^{\pm} - k^{\pm}| \geqq \mathcal{K}_3^{-1}. \tag{124}$$

From estimate (123) we have $\|k^{\pm}\|_{L_x^\infty} \leqq \mathcal{K}_1$. By differentiating $\chi(k^{\pm})$ with respect to x we obtain

$$-\frac{1}{2}k_x^{\pm}\chi'(k^{\pm}) = \int_\Gamma \frac{v_x(\tau)\gamma(\tau)}{(\tau - k^{\pm})^2}\mathrm{d}\tau. \tag{125}$$

Using estimates (124) and equation (125) we get

$$\begin{aligned}
|k_x^{\pm}| &\leqq 2\mathcal{K}_2 \left(\|v_x^+\|_{L_{xa}^\infty} + \|v_x^-\|_{L_{xa}^\infty}\right)\left(\|\gamma^+\|_{L_{xa}^\infty} + \|\gamma^-\|_{L_{xa}^\infty}\right)\left|\int_\Gamma \frac{1}{(\tau - k^{\pm})^2}\mathrm{d}\tau\right| \\
&\leqq 2\mathcal{K}_2 \left(\|v_x^+\|_{L_{xa}^\infty} + \|v_x^-\|_{L_{xa}^\infty}\right)\left(\|\gamma^+\|_{L_{xa}^\infty} + \|\gamma^-\|_{L_{xa}^\infty}\right)\left|\frac{v_b^+ - v_b^-}{(v_b^+ - k^{\pm})(v_b^- - k^{\pm})}\right| \\
&\leqq 4\mathcal{K}_2 \left(\|v_x^+\|_{L_{xa}^\infty} + \|v_x^-\|_{L_{xa}^\infty}\right)\left(\|\gamma^+\|_{L_{xa}^\infty} + \|\gamma^-\|_{L_{xa}^\infty}\right)\frac{1}{|v_b^{\pm} - k^{\pm}|} \\
&\leqq 4\mathcal{K}_2\mathcal{K}_3 \left(\|v_x^+\|_{L_{xa}^\infty} + \|v_x^-\|_{L_{xa}^\infty}\right)\left(\|\gamma^+\|_{L_{xa}^\infty} + \|\gamma^-\|_{L_{xa}^\infty}\right),
\end{aligned}$$

and thus

$$\|k_x^{\pm}\|_{L_x^\infty} \leqq 4\mathcal{K}_2\mathcal{K}_3 \left(\|v_x^+\|_{L_{xa}^\infty} + \|v_x^-\|_{L_{xa}^\infty}\right)\left(\|\gamma^+\|_{L_{xa}^\infty} + \|\gamma^-\|_{L_{xa}^\infty}\right). \tag{126}$$

4.5. The final a priori estimate

Let us define the norm $\overset{\circ}{H}{}^3_{xa}$ such that for all $\varphi(x, a) \in \overset{\circ}{H}{}^3_{xa}$,

$$\begin{aligned}
\|\varphi\|^2_{\overset{\circ}{H}{}^3_{xa}} = {}& \|\varphi\|^2_{L^2_{xa}} + \|\varphi_x\|^2_{L^2_{xa}} + \|\varphi_a\|^2_{L^2_{xa}} + \|\varphi_{xa}\|^2_{L^2_{xa}} + \|\varphi_{xx}\|^2_{L^2_{xa}} \\
&+ \|\varphi_{aa}\|^2_{L^2_{xa}} + \|\varphi_{xxx}\|^2_{L^2_{xa}} + \|\varphi_{xxa}\|^2_{L^2_{xa}} + \|\varphi_{xaa}\|^2_{L^2_{xa}}.
\end{aligned}$$

Let us now set

$$\begin{aligned}
Z(t) = {}& \|\mathcal{R}^+\|^2_{\overset{\circ}{H}{}^3_{xa}} + \|\mathcal{R}^-\|^2_{\overset{\circ}{H}{}^3_{xa}} + \|r^+\|^2_{H^1_x} + \|r^-\|^2_{H^1_x} \\
&+ \|\omega^+\|^2_{H^3_{xa}} + \|\omega^-\|^2_{H^3_{xa}} + \|\gamma^+\|^2_{H^3_{xa}} + \|\gamma^-\|^2_{H^3_{xa}}.
\end{aligned}$$

Using a priori estimates of Section 4.4 and the Sobolev embeddings $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, $H^{1/2}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$ we get

$$\frac{\mathrm{d}Z}{\mathrm{d}t}(t) \leqq \mathcal{F}(Z(t)), \tag{127}$$

where $\mathcal{F}(Z)$ is a monotonically increasing positive function of Z. By integrating equation (127) in time we get

$$Z(t) \leqq Z(0) + \int_0^t \mathcal{F}(Z(s))\mathrm{d}s. \tag{128}$$

A Gronwall's inequality yields a real number $T > 0$ and a positive function $K(t)$, finite on the interval $[0, T]$, such that

$$Z(t) \leqq K(t), \quad \forall t \in [0, T]. \tag{129}$$

From estimate (129), equations (50)–(51), and (64)–(65) we get

$$r^{\pm} \in L^{\infty}\left([0, T]; H^1_x\right) \cap \mathrm{Lip}\left([0, T]; L^2_x\right), \tag{130}$$

$$\mathcal{R}^{\pm} \in L^{\infty}\left([0, T]; \mathring{H}^3_{xa}\right) \cap \mathrm{Lip}\left([0, T]; H^2_{xa}\right), \tag{131}$$

$$\omega^{\pm} \in L^{\infty}\left([0, T]; H^3_{xa}\right) \cap \mathrm{Lip}\left([0, T]; H^2_{xa}\right), \tag{132}$$

$$\gamma^{\pm} \in L^{\infty}\left([0, T]; H^3_{xa}\right) \cap \mathrm{Lip}\left([0, T]; H^2_{xa}\right), \tag{133}$$

$$v^{\pm} \in L^{\infty}\left([0, T]; H^3_{xa}\right), \tag{134}$$

$$k^{\pm} \in L^{\infty}\left([0, T]; W^{1,\infty}_x\right). \tag{135}$$

From estimate (134) and equations (10)–(11) we get

$$v^{\pm} \in L^{\infty}\left([0, T]; H^3_{xa}\right) \cap \mathrm{Lip}\left([0, T]; H^2_{xa}\right), \tag{136}$$

$$\phi \in L^{\infty}\left([0, T]; H^3_x\right) \cap \mathrm{Lip}\left([0, T]; H^2_x\right). \tag{137}$$

Using the bounds (130)–(137) on the solution, the existence and uniqueness of a classical solution are proved by a standard scheme. In fact, the bounds (130)–(136) yield the existence of solution subsequences which converge weakly to a weak limit point solution satisfying estimates (130)–(136). Then, standard compactness arguments (compact Sobolev embeddings, Ascoli's theorem, interpolation inequalities) can be used to pass to the limit in the equations and get local existence of a classical solution. Uniqueness is obtained by considering two different solutions of the equations. We next construct equations for the difference of both solutions by subtracting the original equations of each solution. Then, using the same kind of a priori estimates developed in Section 4.4, where the estimates now concern the difference of both solutions, we get a differential inequality on the L^2-norm of the difference of both solutions. Here again, a Gronwall's inequality shows that two different solutions are equal in L^2 for any time if it holds at the initial time, which gives the uniqueness of the solution.

To summarize, we have the existence and uniqueness theorem.

Theorem 1. *Let us assume that at the initial time,* $r^{\pm}(t=0) \in H_x^1$, $\mathcal{R}^{\pm}(t=0) \in \overset{\circ}{H}{}^3_{xa}$, $\omega^{\pm}(t=0) \in H^3_{xa}$, $\gamma^{\pm}(t=0) \in H^3_{xa}$, $v^{\pm}(t=0) \in H^3_{xa}$. *Let us suppose that* $v^- \leqq v^+$, $\partial_a v^+ < 0$, $\partial_a v^- > 0$, *and* $\partial_x^\alpha v^+(a=1) = \partial_x^\alpha v^-(a=1), \forall \alpha \leqq 1$ *(see Theorem 2). Let us assume that the conditions*

$$\text{ess} \inf_{z\in\Gamma} |\chi^{\pm}(z)| > 0, \quad \textit{and} \quad \text{Ind}(\mathcal{G}) = 2,$$

hold, with $\mathcal{G} = \chi^+/\chi^-$*, and where* $\chi^{\pm}$ *are given by (42). Then the system of integrodifferential equations (10)–(11), (13)–(14) and (50)–(51) are equivalent to each other, and are hyperbolic with real eigenvalues* $\{v^{\pm}, k^{\pm}\}$ *satisfying for all* $x \in \mathbb{R}/\mathbb{Z}$ *and* $t \in [0, T]$,

$$\begin{aligned}
&\chi(k^{\pm}(t,x)) = 0, \quad |\chi'(k^{\pm}(t,x))| > 0,\\
&-\infty < k^-(t,x) < \inf_{a\in[0,1]} v^-(t,x,a) \quad \textit{and} \quad \sup_{a\in[0,1]} v^+(t,x,a) < k^+(t,x) < +\infty,
\end{aligned}$$

where the function $\chi(z)$ *is given by (30). Moreover, there exists a time* $T > 0$ *such that the systems (50)–(51), (10)–(11), (13)–(14), and (64)–(65) have unique solutions*

$$\begin{aligned}
r^{\pm} &\in L^\infty\left([0,T]; H_x^1\right) \cap \text{Lip}\left([0,T]; L_x^2\right),\\
\mathcal{R}^{\pm} &\in L^\infty\left([0,T]; \overset{\circ}{H}{}^3_{xa}\right) \cap \text{Lip}\left([0,T]; H^2_{xa}\right),\\
v^{\pm} &\in L^\infty\left([0,T]; H^3_{xa}\right) \cap \text{Lip}\left([0,T]; H^2_{xa}\right),\\
\phi &\in L^\infty\left([0,T]; H^3_x\right) \cap \text{Lip}\left([0,T]; H^2_x\right),\\
c &\in L^\infty\left([0,T]; H^3_{xa}\right) \cap \text{Lip}\left([0,T]; H^2\right),\\
u &\in L^\infty\left([0,T]; H^3_{xa}\right) \cap \text{Lip}\left([0,T]; H^2_{xa}\right),\\
\phi &\in L^\infty\left([0,T]; H^3_x\right) \cap \text{Lip}\left([0,T]; H^2_x\right),\\
\omega^{\pm} &\in L^\infty\left([0,T]; H^3_{xa}\right) \cap \text{Lip}\left([0,T]; H^2_{xa}\right),\\
\gamma^{\pm} &\in L^\infty\left([0,T]; H^3_{xa}\right) \cap \text{Lip}\left([0,T]; H^2_{xa}\right),
\end{aligned}$$

where the Riemann invariants $r^{\pm}$ *and* $\mathcal{R}^{\pm}$ *are given by*

$$\begin{aligned}
r^{\pm}(t,x) &= k^{\pm}(t,x) + \int_0^1 \ln\left|\frac{v^-(t,x,\nu) - k^{\pm}(t,x)}{v^+(t,x,\nu) - k^{\pm}(t,x)}\right| \mathrm{d}\nu,\\
\mathcal{R}^{\pm}(t,x,a) &= v^{\pm}(t,x,a) + \int_0^1 \ln\left|\frac{v^-(t,x,\nu) - v^{\pm}(t,x,a)}{v^+(t,x,\nu) - v^{\pm}(t,x,a)}\right| \mathrm{d}\nu,
\end{aligned}$$

and are respectively constant along the characteristics curves $x^{\pm}(t)$ *and* $x^{a,\pm}(t)$ *defined by*

$$\frac{\mathrm{d}}{\mathrm{d}t} x^{\pm}(t) = k^{\pm}(t, x^{\pm}(t)) \quad \textit{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} x^{a,\pm}(t) = v^{\pm}(t, x^{a,\pm}(t), a).$$

Finally, we establish some order or monotonicity properties satisfied by the solutions of Theorem 1. We have the following theorem.

Theorem 2. *The solutions of Theorem 1 are order preserving in the sense that for any* $a, b \in [0, 1]$ *we have*

$$v_0^{\pm}(\cdot, a) \leqq v_0^{\pm}(\cdot, b) \Longrightarrow v^{\pm}(\cdot, \cdot, a) \leqq v^{\pm}(\cdot, \cdot, b), \tag{138}$$

$$\operatorname{sign}(\omega_0^{\pm}) = \operatorname{sign}(\omega^{\pm}), \tag{139}$$

$$\operatorname{sign}(v_0^{+}(\cdot, a) - v_0^{-}(\cdot, b)) = \operatorname{sign}(v^{+}(\cdot, \cdot, a) - v^{-}(\cdot, \cdot, b)). \tag{140}$$

Proof. The proof is based on the Crandall–Tartar Theorem concerning relations between nonexpansive and order preserving mappings [23]. Let us assume that $v^{\pm} \in L^{\infty}(0, T; W^{1,1}_{xa})$ and set $\widetilde{\omega}^{\pm} = v_1^{\pm} - v_2^{\pm} = v^{\pm}(t, x, a) - v^{\pm}(t, x, b)$, $\widetilde{v}^{\pm} = (v_1^{\pm} + v_2^{\pm})/2 = (v^{\pm}(t, x, a) + v^{\pm}(t, x, b))/2$, $\widetilde{\rho} = v^{+}(t, x, a) - v^{-}(t, x, b)$, and $\widetilde{v} = (v^{+}(t, x, a) + v^{-}(t, x, b))/2$, then using equations (10) we obtain,

$$\partial_t \widetilde{\omega}^{\pm} + \partial_x \left(\widetilde{v}^{\pm} \widetilde{\omega}^{\pm}\right) = 0, \tag{141}$$

$$\partial_t \omega^{\pm} + \partial_x \left(v^{\pm} \omega^{\pm}\right) = 0, \tag{142}$$

$$\partial_t \widetilde{\rho} + \partial_x \left(\widetilde{v} \widetilde{\rho}\right) = 0. \tag{143}$$

Let us treat the case of equation (141) which leads to the property (138). The two other equations (142) and (143), which lead respectively to the properties (139) and (140), can be treated in the same way. Let $\zeta_h \in \mathscr{C}_0^{\infty}(\mathbb{R})$ be a convex regularization of the modulus function which converges uniformly to $|\cdot|$ as $h \longrightarrow 0$ and satisfies $|\zeta_h'| \leqq 1$. If we multiply equation (141) by $\zeta_h'(\widetilde{\omega}^{\pm})$ and integrate variable x in space, using integration by parts we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \zeta_h(\widetilde{\omega}^{\pm}) \mathrm{d}x &= -\int_0^1 \mathrm{d}x\, \zeta_h'(\widetilde{\omega}^{\pm}) \partial_x \left(\widetilde{v}^{\pm}\, \widetilde{\omega}^{\pm}\right) \\ &\leqq -\int_0^1 \mathrm{d}x\, \partial_x \widetilde{v}^{\pm} \int_0^{\widetilde{\omega}^{\pm}} \zeta_h''(s) s \mathrm{d}s \\ &\leqq \epsilon(h) \left(\|v_1^{\pm}\|_{L^{\infty}(0,T;W^{1,1}_{xa})} + \|v_2^{\pm}\|_{L^{\infty}(0,T;W^{1,1}_{xa})} \right), \end{aligned} \tag{144}$$

where

$$\epsilon(h) = C \sup_{v \in \mathbb{R}} \left| \int_0^v \zeta_h''(s) s \mathrm{d}s \right| \overset{h \to 0}{\longrightarrow} 0.$$

Passing to the limit in (144) as $h \longrightarrow 0$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\widetilde{\omega}^{\pm}\|_{L^1_x} \leqq 0, \tag{145}$$

which, after time integration, is equivalent to

$$\|v^{\pm}(t, \cdot, a) - v^{\pm}(t, \cdot, b)\|_{L^1_x} \leqq \|v_0^{\pm}(\cdot, a) - v_0^{\pm}(\cdot, b)\|_{L^1_x}. \tag{146}$$

If we now define the operators $\mathcal{T}^{\pm} : L^1_x \longrightarrow L^1_x$ by $\mathcal{T}^{\pm} v_0^{\pm} = v^{\pm}$, obviously $\mathcal{T}^{\pm}$ are mappings in L^1_x which conserve the integral, and are nonexpansive in L^1_x thanks to property (145) or (146). Therefore, using Proposition 1 of [23], the operators $\mathcal{T}^{\pm}$ are order preserving in the sense that property (138) is satisfied. □

Acknowledgment. The author would like to thank Yann Brenier for introducing him to the work of V.M. Teshukov about infinite dimensional systems of conservation laws and for related fruitful discussions.

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Institut Jean Lamour,
UMR Nancy-Université CNRS 7198,
Université Henri Poincaré,
Bd des Aiguillettes,
B.P. 70239, 54506
Vandoeuvre-lès-Nancy Cedex,
France.
e-mail: Nicolas.Besse@ijl.nancy-universite.fr

and

Institut de Mathématiques Elie Cartan,
UMR Nancy-Université CNRS INRIA 7502,
Université Henri Poincaré,
Bd des Aiguillettes,
B.P. 70239, 54506
Vandoeuvre-lès-Nancy Cedex,
France.
e-mail: Nicolas.Besse@iecn.u-nancy.fr
e-mail: besse@iecn.u-nancy.fr

(*Received November 5, 2008 / Accepted April 10, 2010*)
Published online January 11, 2011 – © Springer-Verlag (2011)